Deviations from Tri-Bimaximal Mixing

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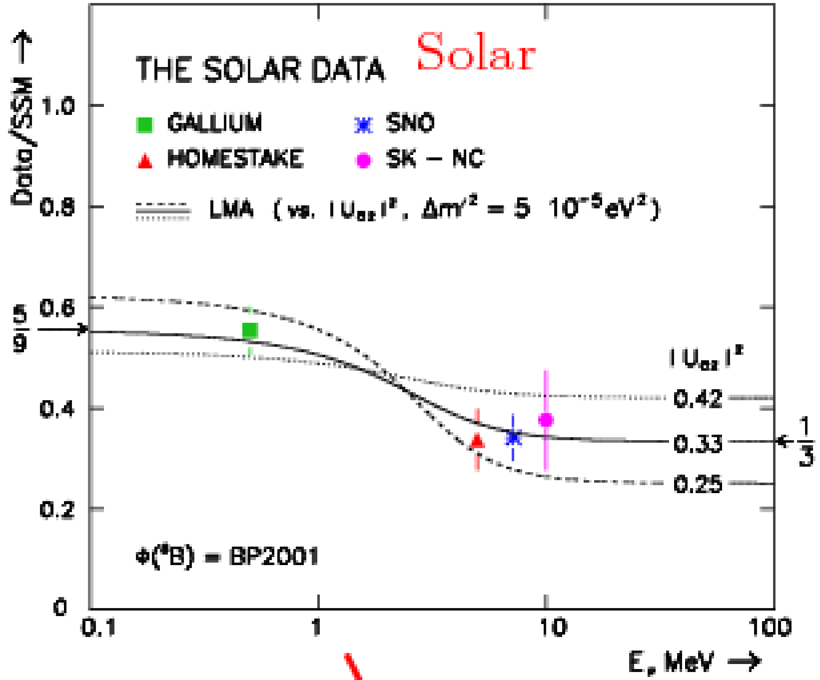
25th April 2006

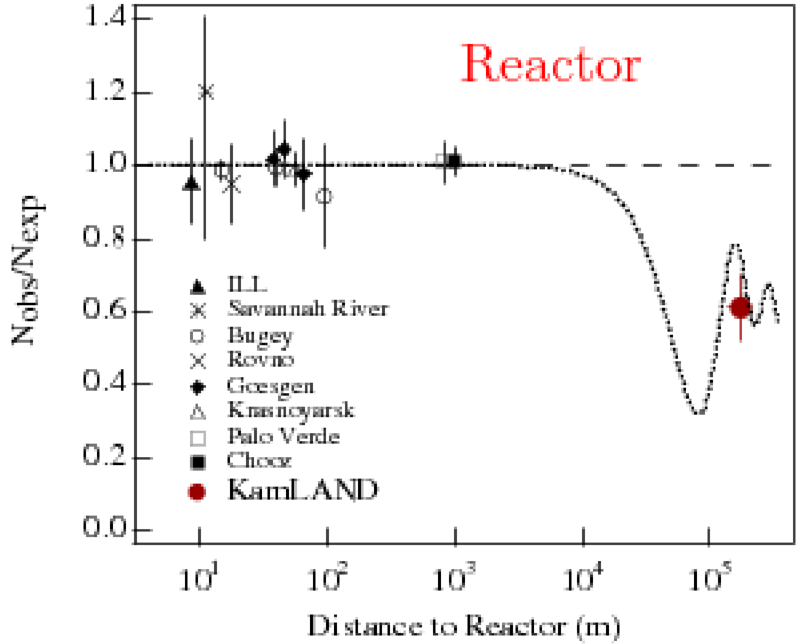
Outline of Talk

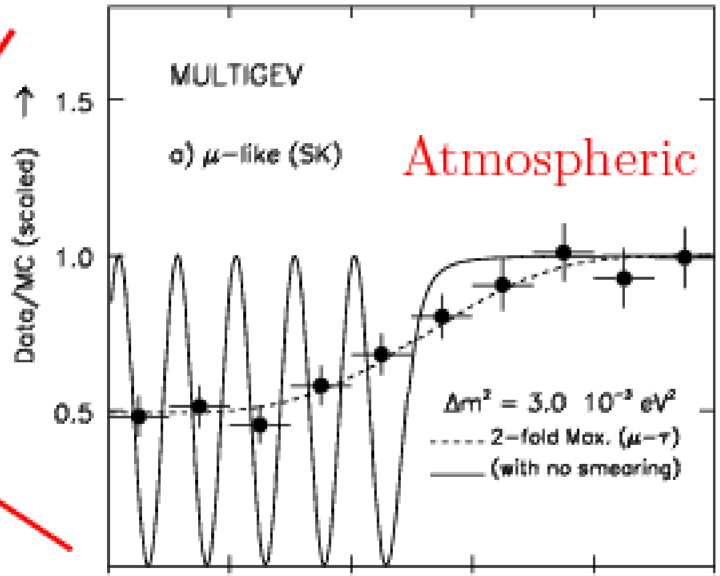
- Introduction to Tri-Bimaximal Mixing (TBM)
- Symmetries of TBM
- Parameterisation of deviations from TBM
- Determination of deviations from TBM
- Discussion

Work done in collaboration with Bill Scott and John Back.

Summary of Neutrino Oscillation Data







$$
(|U|^2) \sim \begin{matrix} \\ e \\ \mu \\ \tau \end{matrix} \begin{matrix} \nu_1 & \nu_2 & \nu_3 \\ \left(\begin{matrix} . & 0.31 \pm 0.04 & < 0.012 \\ . & . & 0.50 \pm 0.11 \\ . & . & . \end{matrix}\right) \end{matrix}
$$

NB. 68% CL!

TriBimaximal Mixing

So, in leading approximation:

$$(|U_{MNS}|^2) \simeq \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix} \end{array} .$$

Corresponding neutrino mass matrix (squared) in the flavour basis, ie.

$M_\nu^2 = U D_\nu^2 U^\dagger$ where $D_\nu^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$, is given by:

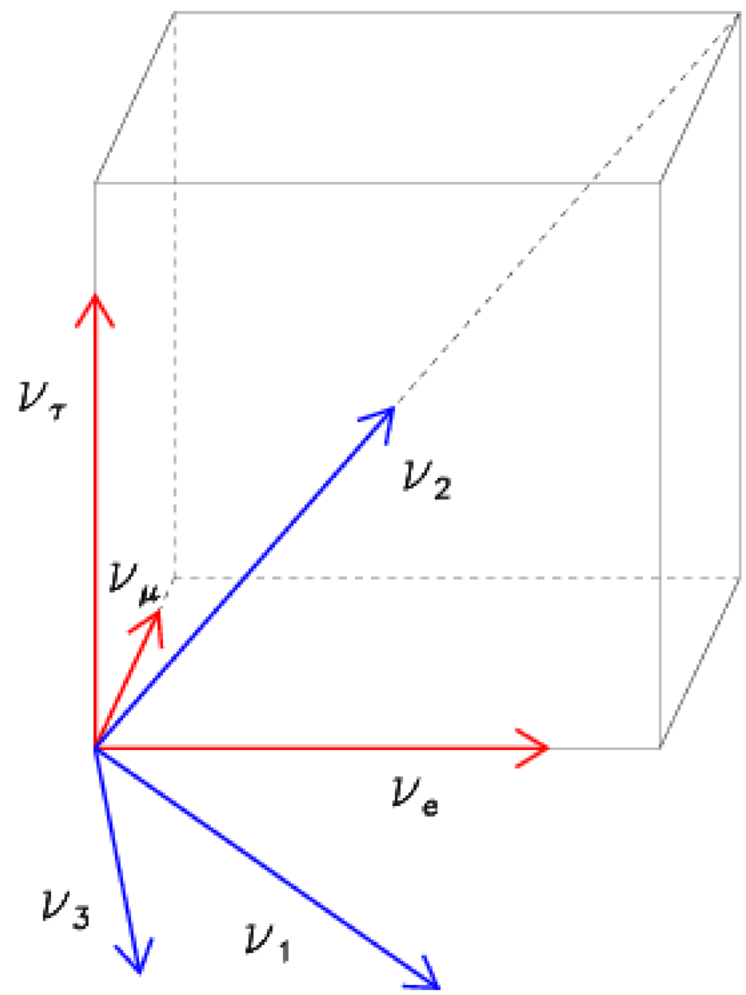
$$M_\nu M_\nu^\dagger = \begin{array}{c} \\ \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_e & \nu_\mu & \nu_\tau \end{array} \\ \begin{pmatrix} s+t+u & u & u \\ u & s+u & t+u \\ u & t+u & s+u \end{pmatrix} \end{array}$$

where $s = (m_1^2 + m_3^2)/2$, $t = -\Delta m_{31}^2/2$ and $u = -\Delta m_{12}^2/3$.

Discussion of TBM

TBM form is redolent of symmetries:

- Taking ν_e, ν_μ, ν_τ to define the orientation of a cube, ν_2 lies along its body diagonal
- Same coefficients form the $M = 0$ subset of the $j \times j = 1 \times 1$ set of Clebsch-Gordan Coeffts.



Several authors have built TBM into models,

eg.

G. Altarelli and F. Feruglio, hep-ph/0512103

I. de Medeiros Varzielas, S. King and G. Ross, hep-ph/0512313

E. Ma PRD 73:057304, 2006

etc.

TBM in Standard Parameterisation

$\exists$ no degrees of freedom in TBM, $\Longrightarrow$ 4 constraints:

$$
\begin{aligned}
\sin\theta_{12} &= \frac{1}{\sqrt{3}} \\
\sin\theta_{23} &= \frac{1}{\sqrt{2}} \\
\mathrm{Re}(U_{e3}) &= 0 \\
\mathrm{Im}(U_{e3}) &= 0
\end{aligned}
$$

Symmetries of TBM: Democracy, $\boldsymbol{CP}$ and $\boldsymbol{\mu-\tau}$ Exchange

Most striking feature is arguably the trimaximally mixed ν_2 mass eigenstate.

Ensured by "Democracy", ie. symmetry of ν mass matrix under transformation:

$$M_\nu^2 \to U_D M_\nu^2 U_D^\dagger \text{ where } U_D = \mathrm{e}^{i\alpha D} \text{ with } D := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

ie. $[M_\nu^2, D] = 0$ (hep-ph/0402006, hep-ph/0403278).

In addition, TBM is defined by two other symmetries: CP and $\mu - \tau$ exchange:

$$[M_\nu^2, P] = 0 \text{ with } P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(hep-ph/0210197).

NB. Decmocracy and $\mu - \tau$-symmetry each implies 2 constraints (see back).

Questions Raised by TBM Ansatz

- Is U_{e3} =0?
- If not, is CP violated?
- Is $\mu - \tau$ symmetry respected?

  ie. is $|U_{\mu 2}| = |U_{\tau 2}|$ and $|U_{\mu 3}| = |U_{\tau 3}|$? Or, in conventional parameterisation, is $\theta_{23} = \pi/4$ and $\delta = \pi/2$?)
- Is Democracy respected?

  ie. is $|U_{e1}|^2 = |U_{e2}|^2 = 1/3$?

Each is, of course, a deviation from TBM.

Deviations from TBM with $U_{e3} \neq 0$ (which respect Democracy) were explored in my talk at Imperial in November last year and in hep-ph/0511201.

Deviation from TBM by Non-zero U_{e3}

Can consider deviation as expansion in small non-zero U_{e3} while maintaining trimaximally mixed ν_2, ie.

$$
\begin{aligned}
U_{MNS} &\simeq \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} C & 0 & \sqrt{\frac{3}{2}}U_{e3} \\ 0 & 1 & 0 \\ -\sqrt{\frac{3}{2}}U_{e3}^* & 0 & C \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{\sqrt{6}}C & \frac{1}{\sqrt{3}} & U_{e3} \\ -\frac{1}{\sqrt{6}}C - \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \\ -\frac{1}{\sqrt{6}}C + \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \end{pmatrix} \\
&\simeq \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & U_{e3} \\ -\frac{1}{\sqrt{6}} + \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{1}{2}U_{e3} \\ -\frac{1}{\sqrt{6}} - \frac{\sqrt{3}}{2}U_{e3}^* & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} - \frac{1}{2}U_{e3} \end{pmatrix}.
\end{aligned}
$$

where $C = \sqrt{1 - \frac{3}{2}|U_{e3}|^2} \simeq 1.$

Symmetries in this Parameterisation

With

$$U_{MNS} = \begin{pmatrix} \frac{2}{\sqrt{6}}C & \frac{1}{\sqrt{3}} & U_{e3} \\ -\frac{1}{\sqrt{6}}C - \frac{\sqrt{3}}{2}U^*_{e3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \\ -\frac{1}{\sqrt{6}}C + \frac{\sqrt{3}}{2}U^*_{e3} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}C - \frac{1}{2}U_{e3} \end{pmatrix}$$

Democracy is respected: $[M_\nu^2, D] = 0$ still.

Obviously, $\mathrm{Im}(U_{e3})$ breaks CP:

$$|J_{CP}| = \frac{C\,\mathrm{Im}(U_{e3})}{3\sqrt{2}}$$

while $\mathrm{Re}(U_{e3})$ breaks $\mu - \tau$ symmetry:

$$|U_{\mu 3}|^2 - |U_{\tau 3}|^2 = \sqrt{2}\,C\,\mathrm{Re}(U_{e3}).$$

NB. With $\mathrm{Re}(U_{e3}) = 0, \mathrm{Im}(U_{e3}) \neq 0$, $\mu - \tau$ symmetry is maintained under simultaneous $\mu - \tau$ exchange and CP transformation - called $\mu - \tau$-reflection.

Problem with Standard Parameterisation

Would like to generalise further, parameterising deviations from Democracy again by expansion in small quantities.

However, although we currently have a trimaximally mixed ν_2 eigenstate, $\sin\theta_{12}$ now given by:

$$\sin\theta_{12} = \frac{1}{\sqrt{3}\cos\theta_{13}}.$$

Already differs from $1/\sqrt{3}$ by a (second order) small quantity!

This is a problem (at least as long as we don't know θ_{13}) - small deviations from Democracy are no longer parameterised by $|\sin\theta_{12}|^2 - 1/3$.

Would like our parameterisation to "reflect" our symmetries as much as possible.

Solution

Expand PMNS matrix as:

$$U = P_0 O'_{23} O'_{12} P_{\delta'} O'_{13} P^{\dagger}_{\delta'}$$

(where order differs from PDG convention).

- O'_{ij} are orthogonal rotations in the ij plane (through, in general, different rotation angles than the standard parameterisation)
- $P_{\delta'} = \mathrm{diag}(1, 1, \mathrm{e}^{i\delta'})$
- $P_0 = \mathrm{diag}(1, 1, -1)$.

Then TBM given by:

$$\sin\theta_{12}{}' = 1/\sqrt{3}, \quad \sin\theta_{23}{}' = 1/\sqrt{2},$$
$$\sin\theta_{13}{}'\mathrm{e}^{-i\delta'} = 0$$

where first two constraints sufficient to ensure Democracy, and a trimaximally mixed ν_2 mass eigenstate, independent of $\theta_{13}{}'$.

Expansion in Small Quantities Around TBM

Let

$$\theta_{12}' = \sin^{-1}(1/\sqrt{3}) + \sqrt{2}\epsilon, \quad \theta_{23}' = \sin^{-1}(1/\sqrt{2}) + \mu,$$

$$\sin\theta_{13}' e^{-i\delta'} = U_{e3}/\cos\theta_{12}' = (\rho + i\,\eta)/\cos\theta_{12}',$$

NB, $U_{e3} \equiv \rho + i\,\eta$.

Now:

$$U_{MNS} \simeq \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{(zeroth order)}$$

$$+ \begin{pmatrix} . & \frac{1}{\sqrt{3}}\,(2\epsilon) & \rho + i\,\eta \\ . & \frac{1}{\sqrt{3}}(-\epsilon - \mu) & +\frac{1}{\sqrt{2}}(+\mu - \frac{(\rho + i\,\eta)}{\sqrt{2}}) \\ . & \frac{1}{\sqrt{3}}(-\epsilon + \mu) & -\frac{1}{\sqrt{2}}(-\mu + \frac{(\rho + i\,\eta)}{\sqrt{2}}) \end{pmatrix} \quad \text{(first order)}$$

$$+ \quad \text{higher order terms}$$

Interpretation of Parameters

$\epsilon = \mu = \rho = \eta = 0 \implies$ TBM

$\epsilon = \mu = 0 \implies$ Democracy (independent of ρ and η, ie. U_{e3})

$\mu = \rho = 0 \implies$ $\mu - \tau$ symmetry (independent of ϵ and η)

$\eta = 0 \implies$ CP symmetry etc.

Summary:

Parameter	CP	Democracy	$\mu - \tau$ Reflection
ρ	Y	Y	N
η	N	Y	Y
ϵ	Y	N	Y
μ	Y	N	N

Table 1: Indicates which expansion parameters respect which of three phenomenological symmetries defining TBM.

NB: None of analogous small parameters in PDG convention respects Democracy.

Determination of Deviations from TBM

Naturally, would like to determine ϵ, μ, ρ and η by experiment

Need oscillation probabilities in terms of them in order to see where sensitive.

First, need to consider other small quantities in problem, ie. Δm^2_{21} and the experimental uncertainty in Δm^2_{31}.

Define

$$\Delta \equiv < \Delta m^2_{31} > L/4E$$

with $< \Delta m^2_{31} >$ the (known) World average value of Δm^2_{31}.

Then the two small quantities

$$h \equiv \frac{\Delta m^2_{21}}{< \Delta m^2_{31} >} \simeq 0.03 \quad \text{and} \quad \delta_{31} \equiv \frac{\Delta m^2_{31} - < \Delta m^2_{31} >}{< \Delta m^2_{31} >}$$

describe respectively the hierarchy of mass-squared differences (known to reasonable precision), and the error in the world average value of Δm^2_{31} (not known, by definition - would also like to determine).

NB. Will also use: $S = \sin\Delta$ and $C = \cos\Delta$, known exactly.

Disappearance Probabilities (in vacuum)

$$\frac{1}{4}(1 - P(\nu_e \to \nu_e)) = S^2(\rho^2 + \eta^2) + \frac{2}{9}\Delta^2 h^2 \qquad \text{(2nd order)}$$

$$+ \ 2\Delta SC(\rho^2 + \eta^2)(\delta_{31} - \frac{1}{3}h) + \frac{4}{9}\Delta^2 h^2 \epsilon \qquad \text{(3rd order)}$$

$$\frac{1}{4}(1 - P(\nu_\mu \to \nu_\mu)) = \frac{1}{4}S^2 + \Delta SC(\frac{\delta_{31}}{2} - \frac{h}{3}) \qquad \text{(0th \& 1st order)}$$

$$- \ S^2(\mu - \frac{\rho}{\sqrt{2}})^2 + \frac{2}{3}\Delta SCh(\epsilon + \frac{\rho}{\sqrt{2}})$$

$$+ \ \Delta^2[(1 - 2S^2)\delta_{31}(\frac{\delta_{31}}{4} - \frac{h}{3}) + (2 - 3S^2)\frac{h^2}{9}] \qquad \text{(2nd order)}$$

As is well-known, $P(\nu_e \to \nu_e)$ allows the extraction of $|U_{e3}|^2 = \rho^2 + \eta^2$.

$P(\nu_\mu \to \nu_\mu)$ is sensitive at 1st order to δ_{31}, ie. Δm^2_{13}, with uncertainties due to deviations from TBM entering only at 2nd order.

Presumably why MINOS' first measurement was Δm^2_{13}.

Appearance Probabilities (in vacuum)

$$
\begin{aligned}
\frac{1}{4}P(\nu_e \to \nu_\mu) &= \frac{1}{2}S^2(\rho^2+\eta^2) + \Delta S\frac{\sqrt{2}h}{3}(\rho C + \eta S) + \Delta^2\frac{h^2}{9} \qquad \text{(2nd order)} \\
&+ S^2(\rho^2+\eta^2)(\mu - \frac{\rho}{\sqrt{2}}) \\
&+ \Delta S[(\delta_{31} - \frac{h}{3})(\rho^2+\eta^2)C + \frac{\sqrt{2}h\epsilon}{3}(\rho C + \eta S)] \\
&+ \Delta^2\frac{\sqrt{2}h}{3}[\frac{\sqrt{2}h}{3}(\epsilon - \mu) + \delta_{31}\rho - (2\delta_{31} - h)S(\rho S - \eta C)] \quad \text{(3rd order)} \\
\frac{1}{4}P(\nu_e \to \nu_\tau) &= \frac{1}{2}S^2(\rho^2+\eta^2) - \Delta S\frac{\sqrt{2}h}{3}(\rho C + \eta S) + \Delta^2\frac{h^2}{9} \qquad \text{(2nd order)} \\
&- S^2(\rho^2+\eta^2)(\mu - \frac{\rho}{\sqrt{2}}) \\
&+ \Delta S[(\delta_{31} - \frac{h}{3})(\rho^2+\eta^2)C - \frac{\sqrt{2}h\epsilon}{3}(\rho C + \eta S)] \\
&+ \Delta^2\frac{\sqrt{2}h}{3}[\frac{\sqrt{2}h}{3}(\epsilon + \mu) - \delta_{31}\rho + (2\delta_{31} - h)S(\rho S - \eta C)] \quad \text{(3rd order)}
\end{aligned}
$$

Appearance Probabilities (in vacuum, contd.)

$$\begin{aligned}\frac{1}{4}(P(\nu_\mu \to \nu_\tau)) &= \frac{1}{4}S^2 + \Delta SC(\frac{\delta_{31}}{2} - \frac{h}{3}) \qquad \text{(0th \& 1st order)}\\ &- S^2[(\mu - \frac{\rho}{\sqrt{2}})^2 + \frac{1}{2}(\rho^2 + \eta^2)] + \frac{2}{3}\Delta Sh(\epsilon C + \frac{\eta}{\sqrt{2}}S)\\ &+ \Delta^2[(1-2S^2)\delta_{31}(\frac{\delta_{31}}{4} - \frac{h}{3}) + (1-3S^2)\frac{h^2}{9}] \quad \text{(2nd order)}\end{aligned}$$

Didn't consider remaining oscillation probabilities, as they each start with a ν_τ. All are anyway trivially calculable from above by unitarity.

Discussion

All the usual terms are there for $P(\nu_e \to \nu_\mu)$.

$\mu - \tau$ reflection symmetry is explicit between $P(\nu_e \to \nu_\mu)$ and $P(\nu_e \to \nu_\tau)$, when $\rho = 0$, $\mu = 0$.

Interesting combinations can be made, eg.

$P(\nu_e \to \nu_\mu) - P(\nu_e \to \nu_\tau)$ which depends only on ρ and η at lowest order (with μ entering at higher orders).

$P(\nu_\mu \to \nu_e) - P(\nu_e \to \nu_\tau)$, depends only on ρ at lowest order.

μ and ϵ are, unfortunately, rather well hidden, never entering at leading order, and will be hard to measure.

Especially true for ϵ, as it is always suppressed by a factor of at least $h = 0.03$!

Although work done without matter effects, so far, we know, for example, that $\mu - \tau$ symmetry is respected by matter effects, so some results will survive.

Spare Slide: Small Quantities Expansion in PDG Parameterisation

In PDG Parameterisation, PMNS matrix is:

$$U = O_{23} P_\delta O_{13} P_\delta^\dagger O_{12}$$

Small quantities expansion:

$$\theta_{12} = \sin^{-1}(1/\sqrt{3}) + \sqrt{2}\epsilon, \quad \theta_{23} = \sin^{-1}(1/\sqrt{2}) + \mu,$$

$$\sin\theta_{13} e^{-i\delta} = U_{e3} = (\rho + i\eta),$$

Symmetries:

Parameter	CP	Democracy	$\mu - \tau$ Reflection
ρ	Y	N	N
η	N	N	Y
ϵ	Y	N	Y
μ	Y	N	N

Table 2: Which expansion parameters respect which symmetries (PDG Param.)