Apparent Emittance Growth from Weak Relativistic Effects

Consider a beam in a drift region with zero transverse emittance. Then for an initial distribution centered about $z = 0$, $p_z = p_R$ where p_R is also the reference momentum,

$$\frac{dp_z}{dt} = 0$$

$$\frac{dz}{dt} = v - v_R = c\left[f(p_z/mc) - f(p_R/mc)\right]$$

where

$$f(x) = \frac{x}{\sqrt{1+x^2}}$$

Expanding in $u \equiv (p_z - p_R)/mc$, we have

$$\begin{aligned} z =& z_0 + t\Delta v = \\ & z_0 + \frac{ctu}{\gamma_R^3}\Big[1 - u\frac{3p_R}{2mc\gamma_R^2} + u^2\frac{1}{2\gamma_R^4}\left(4\frac{p_R^2}{m^2c^2} - 1\right) \\ & + u^3\frac{p_R}{8mc\gamma_R^6}\left(11 - \frac{24p_R^2}{m^2c^2}\right) + \ldots\Big] \end{aligned}$$

where $z = z_0$ at $t = 0$ and $\gamma_R^2 = 1 + p_R^2/m^2c^2$.

Look at first corrections to longitudinal emittance: take ϵ_L^2 to order σ^6, second order moments to order σ^4 and first order moments to order σ^2.

Clearly, $< p_z >= p_R$ and $< p_z^2 >= p_R^2 + \sigma_p^2$, because momentum is a constant of the motion.

Using Liouville theorem, distribution function $F(z, p_z) = F(z_0, p_z)$ is also a constant of the motion, so

$$< G(z, p_z) >_t = < G(z_0 + t\Delta v, p_z) >_0 \ .$$

Thus, using $< z_0 >= 0$, $< z_0^2 >= \sigma_z^2$, $< u >= 0$ and $< u^2 >= \sigma_p^2/m^2c^2$, we have

$$< z >=< z_0 > + < t\Delta v >= -\frac{ct}{\gamma_R^3}\frac{\sigma_p^2}{m^2c^2}\frac{3p_R}{2mc\gamma_R^2} + ...$$

$$\begin{aligned}< z^2 > &= < z_0^2 > +2 < z_0 t\Delta v > + < t^2(\Delta v)^2 > \\ &= \sigma_z^2 + \frac{c^2t^2}{\gamma_R^6}\left[\frac{\sigma_p^2}{m^2c^2} + < u^4 > \frac{1}{\gamma_R^4}\left(\frac{25p_R^2}{4m^2c^2} - 1\right) + ...\right]\end{aligned}$$

$$\begin{aligned}< p_z z > - < p_z >< z > &= < (p_z - p_0)z >= mc < uz > \\ &= \frac{mc^2t}{\gamma_R^3}\left[\frac{\sigma_p^2}{m^2c^2} + < u^4 > \frac{1}{2\gamma_R^4}\left(4\frac{p_R^2}{m^2c^2} - 1\right) + ...\right]\end{aligned}$$

Here we have taken advantage of the fact that Δv is independent of z_0, so there are no hidden cross terms, and also set $< z_0^3 >=< u^3 >= 0$.

To this order,

$$\begin{aligned}< z^2 > - < z >^2 &= \sigma_z^2 + \frac{c^2t^2}{\gamma_R^6}\Big[\frac{\sigma_p^2}{m^2c^2} \\ &+ < u^4 > \frac{1}{\gamma_R^4}\left(\frac{25p_R^2}{4m^2c^2} - 1\right) - \frac{\sigma_p^4}{m^4c^4}\frac{9p_R^2}{4m^2c^2\gamma_R^4}\Big]\end{aligned}$$

so the determinant of the 2D covariant matrix up to order σ^6 is

$$\epsilon_L^2 = \sigma_p^2\left[\sigma_z^2 + \frac{9c^2t^2p_R^2}{4m^2c^2\gamma_R^{10}}\left(< u^4 > - \frac{\sigma_p^4}{m^4c^4}\right)\right]$$

For a gaussian beam, $< u^4 >= 3 < u^2 >^2$ and the emittance growth is given by

$$\epsilon_L^2 = \sigma_p^2\sigma_z^2\left[1 + \frac{t^2}{\tau^2}\right]$$

where

$$\tau = \frac{\sqrt{2}}{3}\frac{m^2c^2\gamma_R^4\sigma_z}{v_R\sigma_p^2}$$

or in terms of distance travelled

$$v_R \tau = \frac{\sqrt{2}}{3} \frac{\gamma_R^2}{\beta_R^2} \frac{\sigma_z p_R^2}{\sigma_p^2}$$

It is apparent that for either very relativistic or very unrelativistic beams, this effect becomes small. For a water bag model, the only difference is that $< u^4 >= 2 < u^2 >^2$, so the factor $\sqrt{2}/3$ is replaced by $2/3$.

A more general form for the moments, without trying to simplify is

$$< p >= p_R, \qquad < z >= \frac{s}{v_R}(< v(p) > -v_R)$$

$$< p^2 > - < p >^2 = \sigma_p^2$$

$$< z^2 > - < z >^2 = \sigma_z^2 + \frac{s^2}{v_R^2}\left[< v^2(p) > - < v(p) >^2\right] + \frac{2s}{v_R} < z_0 v(p) >$$

$$< pz > - < p >< z >=< pz_0 > + \frac{s}{v_R}\left[< pv(p) > -p_R < v(p) >\right]$$

If $< z_0 p >=< z_0 v >= 0$, then

$$\epsilon_L^2 = \sigma_p^2 \sigma_{z0}^2 + \frac{s^2}{v_R^2}\Big[\sigma_p^2(< v^2(p) > - < v(p) >^2) - (< pv(p) > -p_R < v(p) >)^2\Big]$$

still grows quadratically in time.