FFAG's Wonderful World of Nonlinear Longitudinal Dynamics

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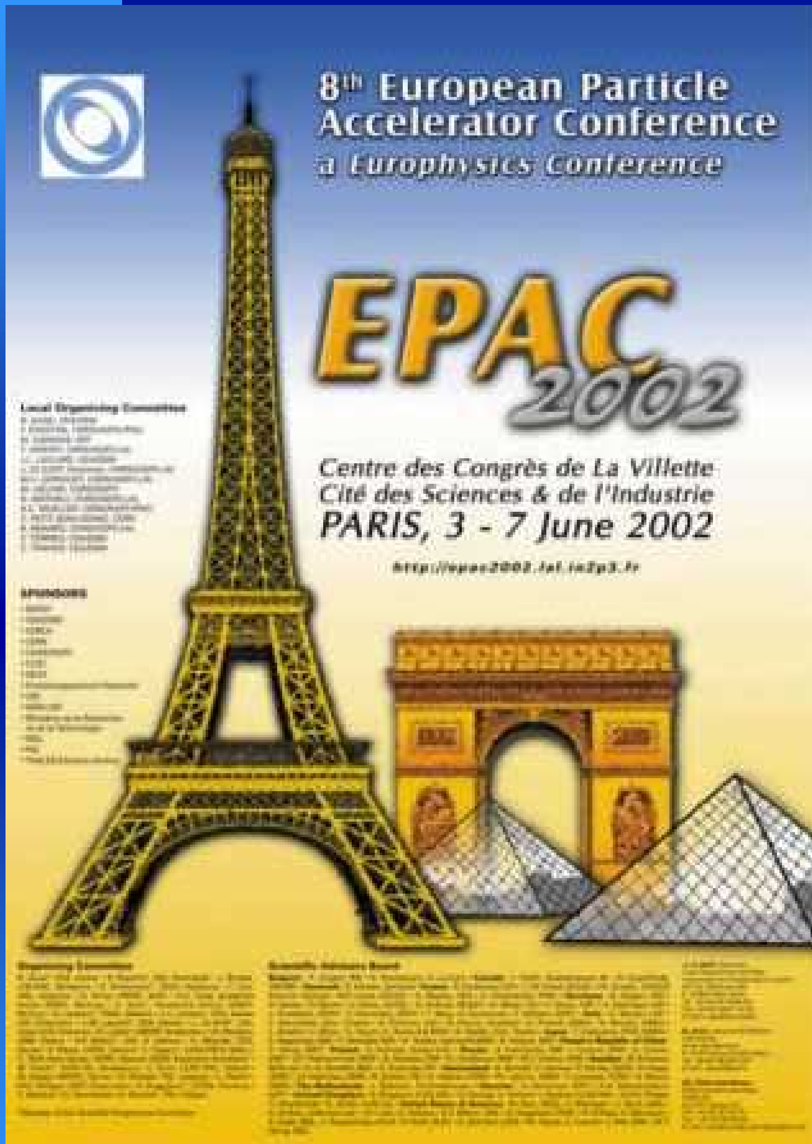
- Phase space properties of **pendula** with nonlinear dependence of 'speed' on momentum
  – characterized by discontinuous behaviour w.r.t. parameters
- F0D0 or regular triplet FFAG lattices lead to quadratic $\Delta L(p)$
  - gutter acceleration when energy gain/cell exceeds critical value
- Asynchronous acceleration:
- Normal mode: rf commensurate with revolution f @ fixed points
  Fundamental & harmonics
- Slip mode: rf deviates from revolution frequency @ fixed points
  Fundamental & harmonics
- Conclusions and outlook



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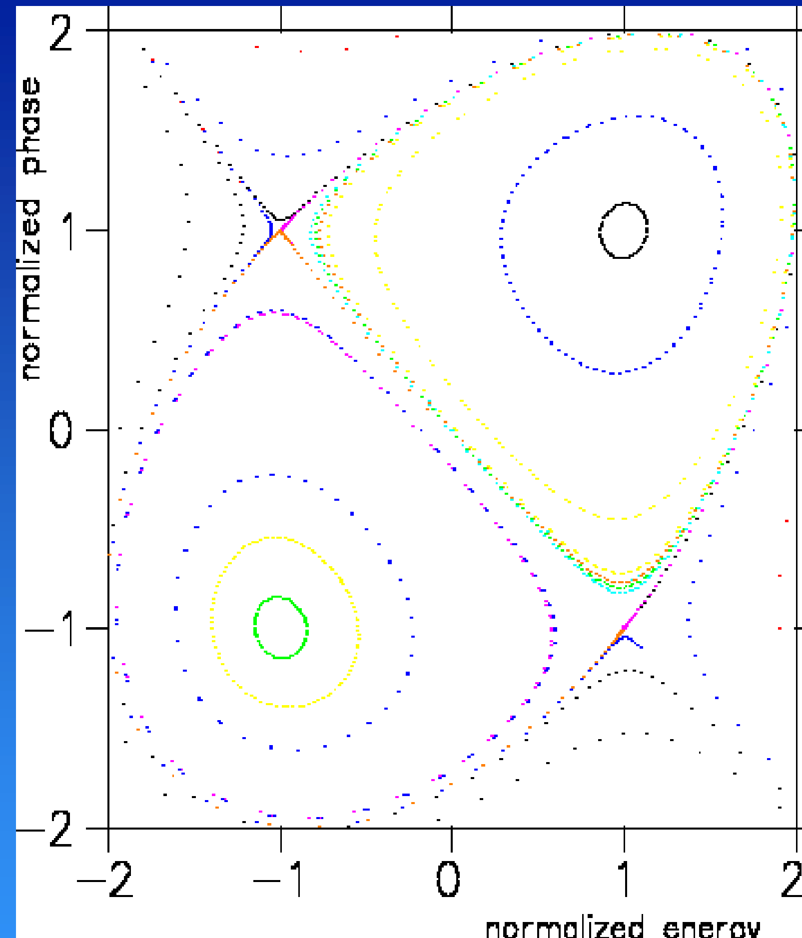


Phase space of the equations
x'=(1-y²) and y'=(x² –1)

Linear Pendulum Oscillator

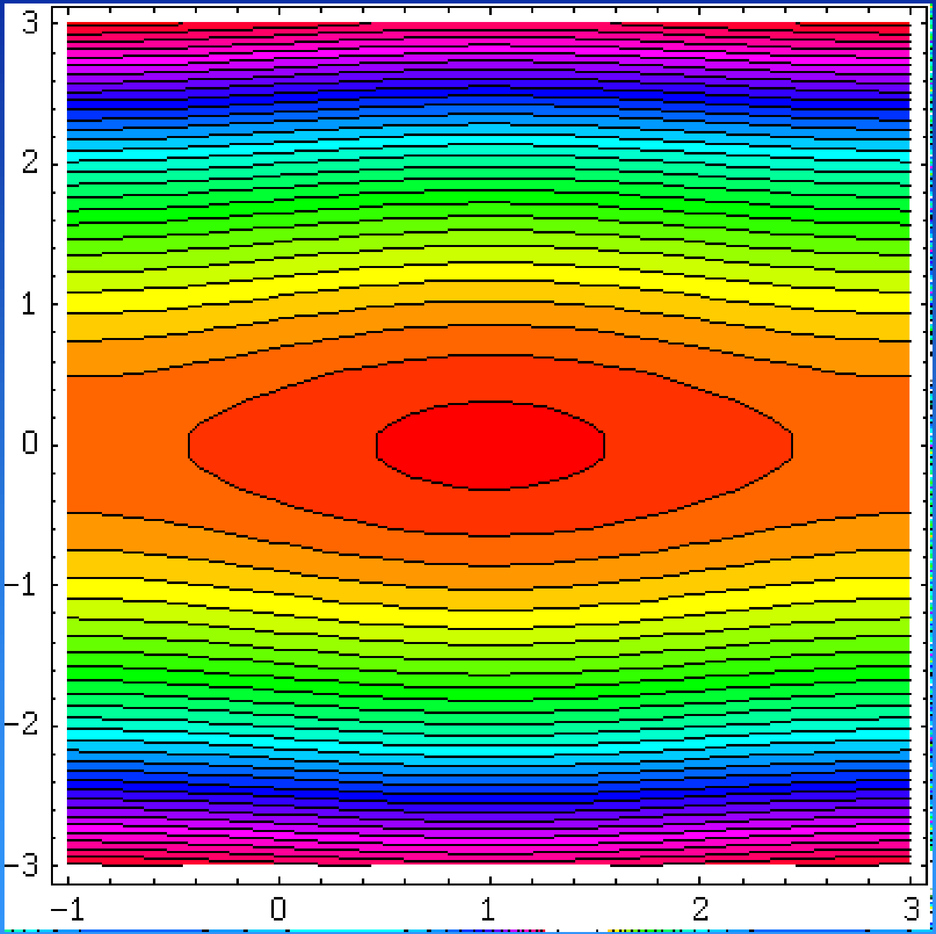
Manifold: set of phase-space paths delimited by a separatrix

Rotation: bounded periodic orbits

Libration: unbounded, possibly semi-periodic, orbits

For simple pendulum, libration paths cannot become connected.

Phase space of the equations
***x'=y* and *y'=a*.Cos(*x*)**



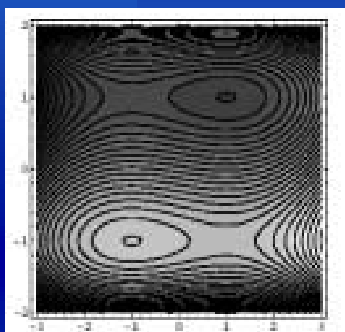
Animation: evolution of phase space as strength `*a*' varies.

Bi-parabolic Oscillator

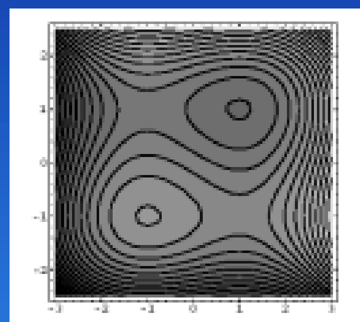
[GIF Animations @ W3]

Topology discontinuous at $a = 1$

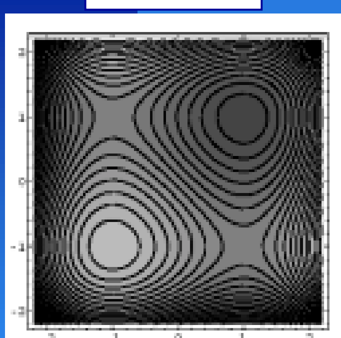
- For $a < 1$ there is a sideways serpentine path
- For $a > 1$ there is a upwards serpentine path
- For $a \equiv 1$ there is a trapping of two counter-rotating eddies within a background flow.



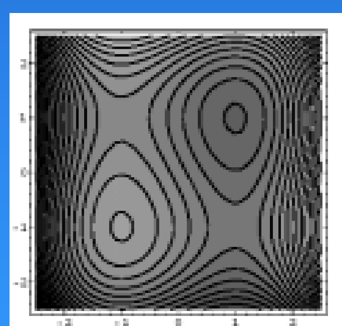
a=1/10



a=1/2



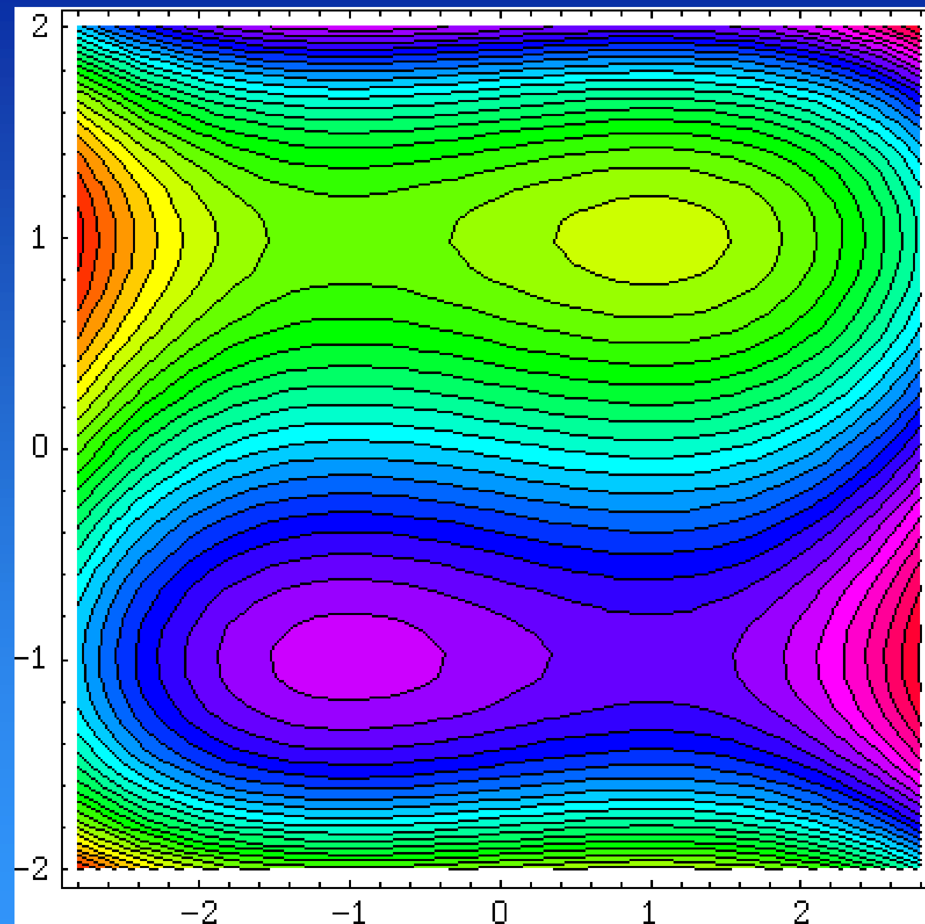
a=1



a=2

Condition for connection of libration paths: $a \geq 1$

Phase space of the equations
$x' = (1-y^2)$ and $y' = a(x^2-1)$



Animation: evolution of phase space as strength `*a*' varies.

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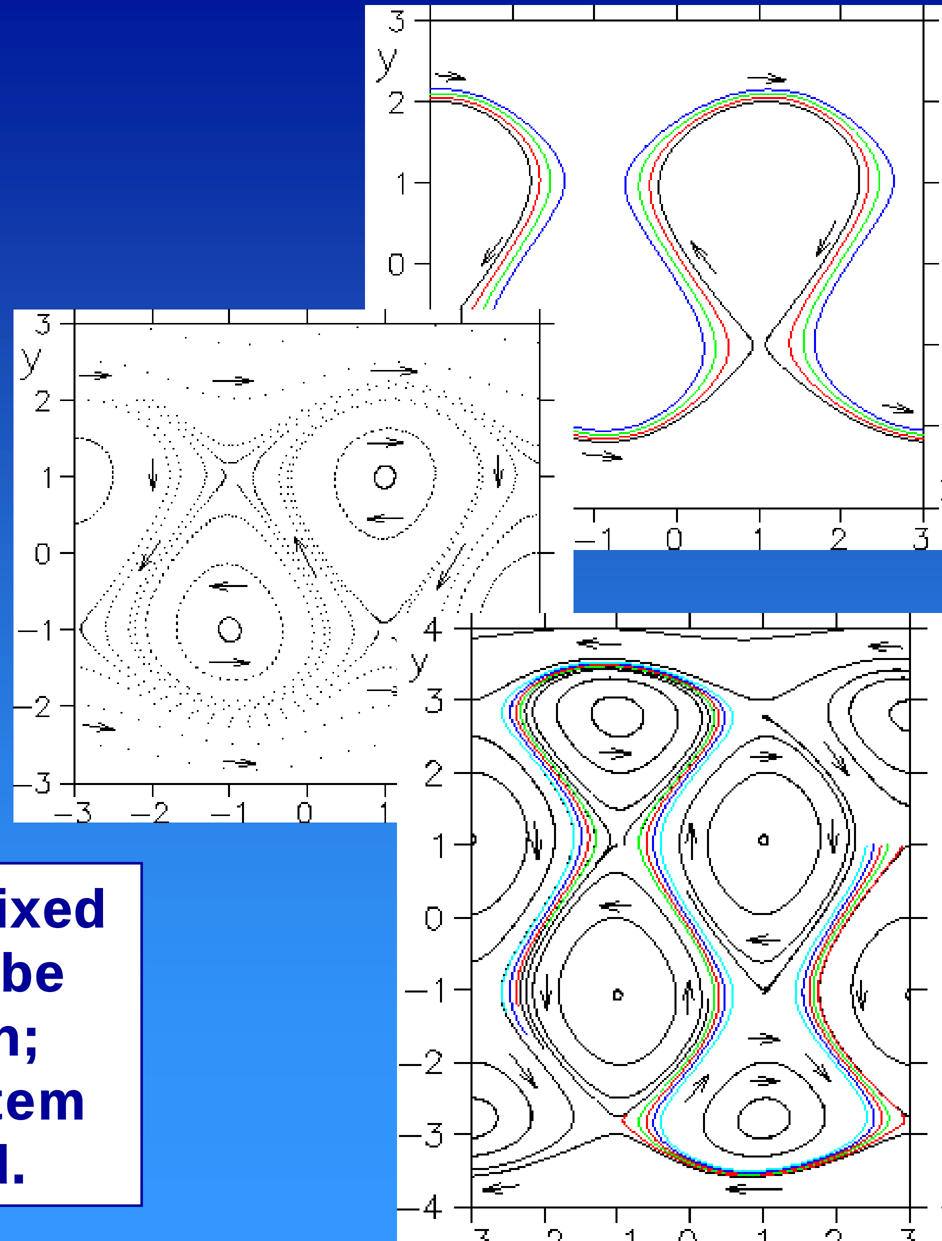
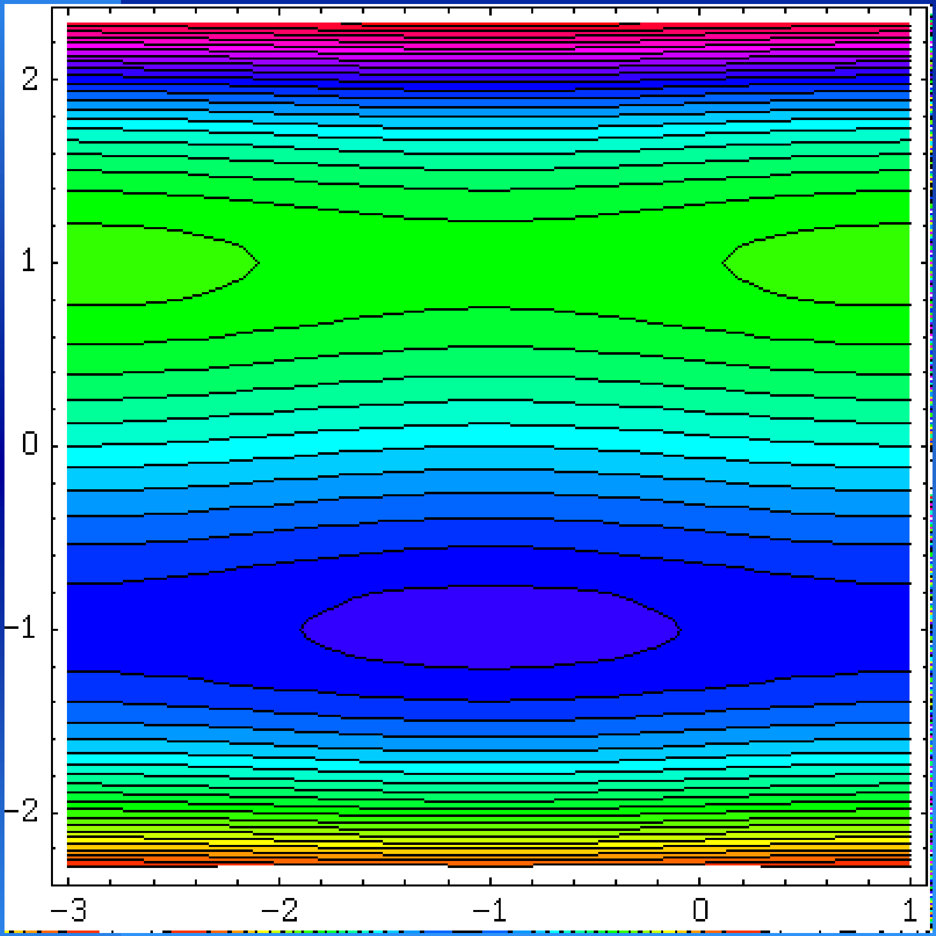


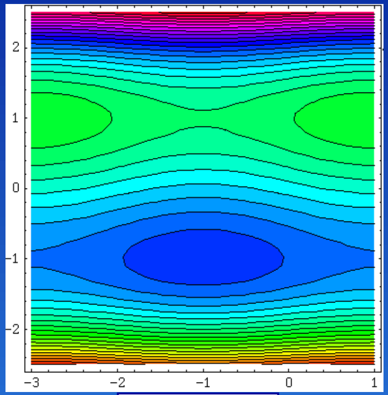
Conditions for connection of fixed points by libration paths may be obtained from the hamiltonian; typically critical values of system parameters must be exceeded.

Quadratic Pendulum Oscillator

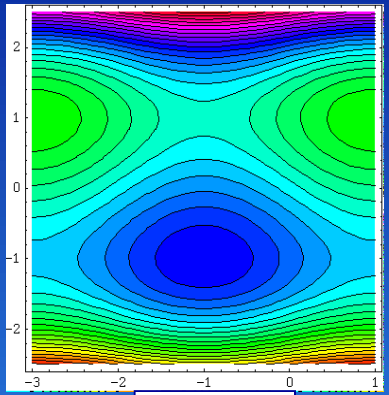


Animation: evolution of phase space as strength `*a*' varies.

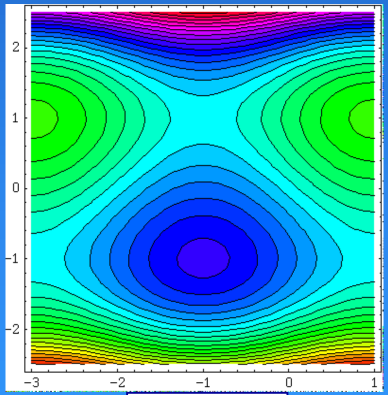
Phase space of the equations
x'=(1-y²)* and *y'=a.*Cos*(x)



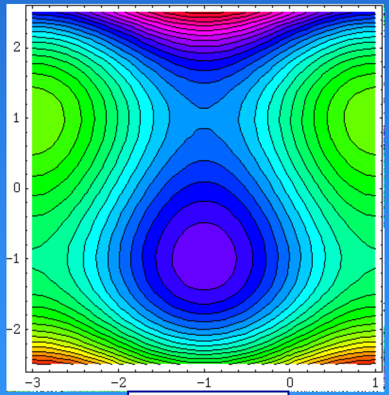
a=1/6



a=1/2



a=1



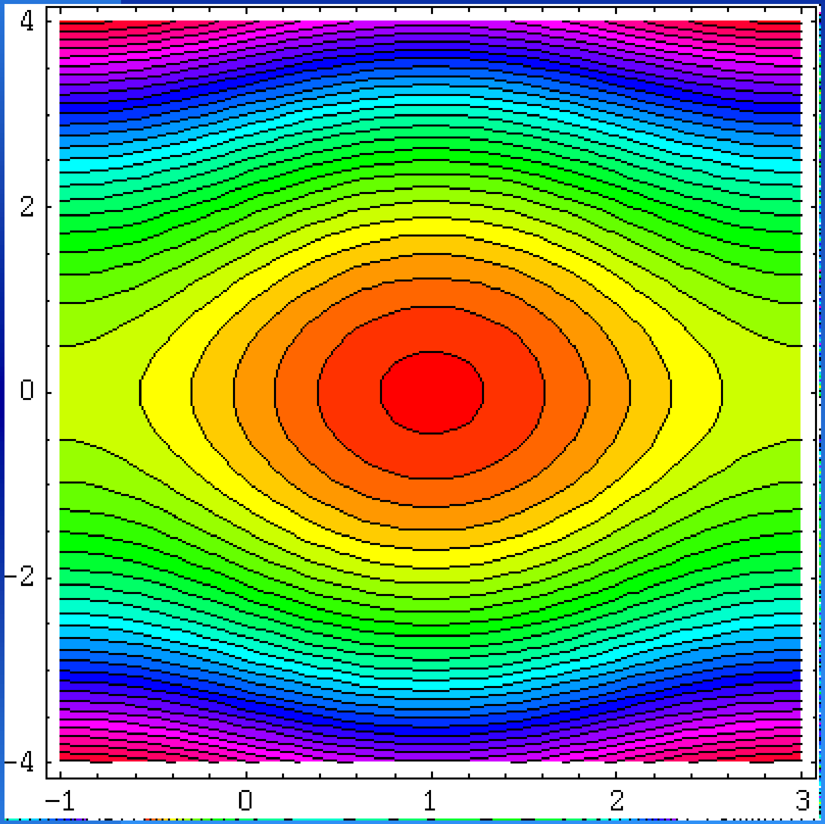
a=2

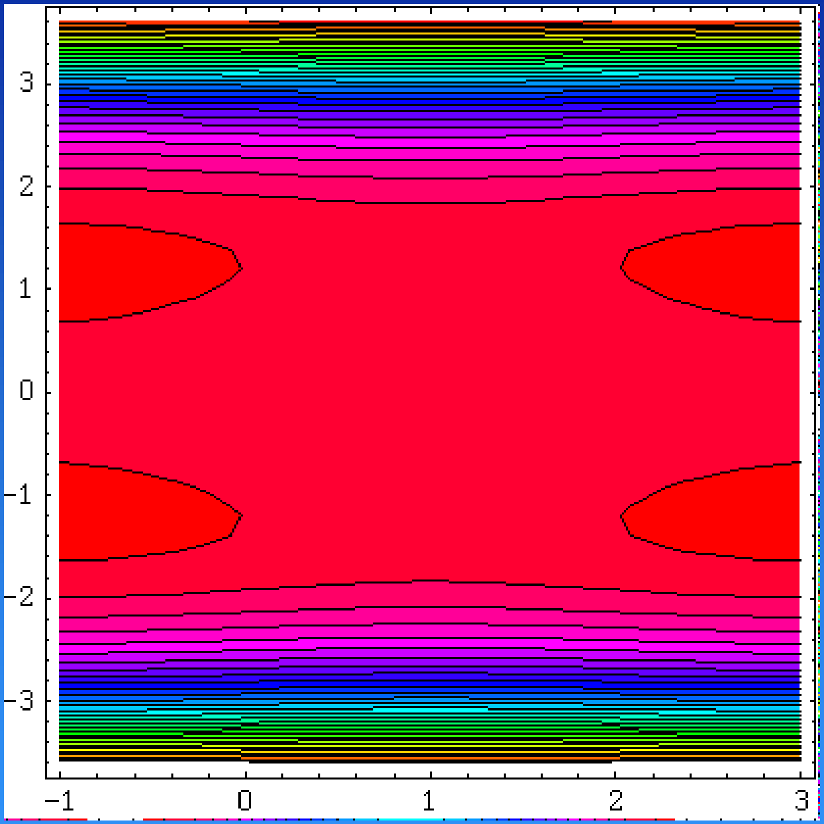
Condition for connection of libration paths: $a \geq 2/3$

Cubic Pendulum Oscillator

Phase space of the equations $x'=y(1-b^2y^2)$ **and** $y'=a.\text{Cos}(x)$

Animations: evolution of phase space as strengths `*a,b*' vary.





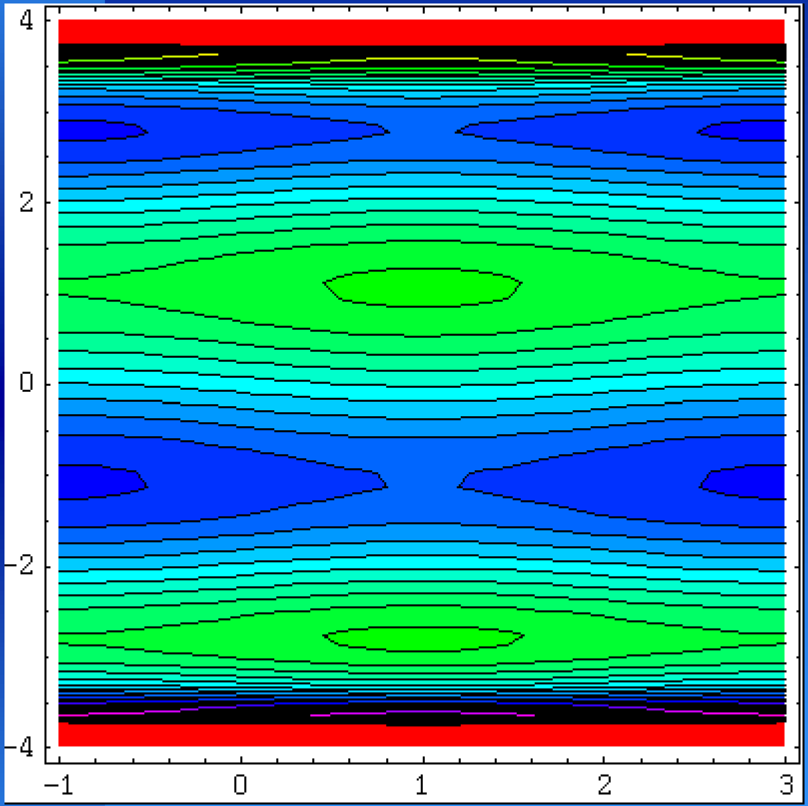
Parameter `*b*` is varied from 0.1 to 1 while `*a*` held fixed at $a=1$.

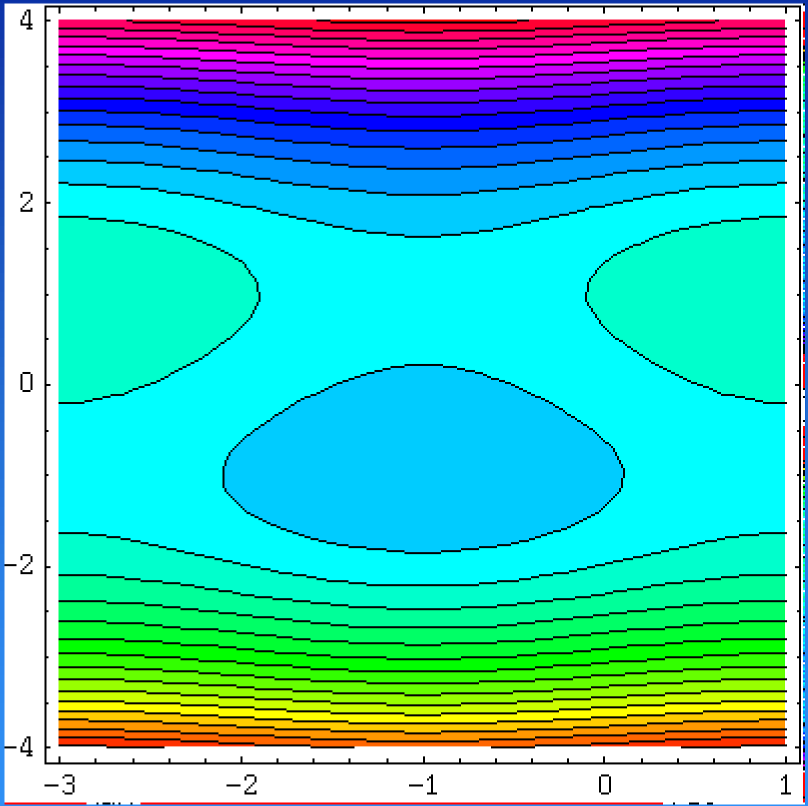
Parameter `*b*` is varied from to 0.8 to 0.14 while `*a*` varies as $a=1/(8b^2)$.

Quartic Pendulum Oscillator

Phase space of the equations
$x' = y^2(1 - b^2 y^2) - 1$
and $y' = a.\text{Cos}(x)$

Animations: evolution of phase space as strengths `*a,b*' vary.





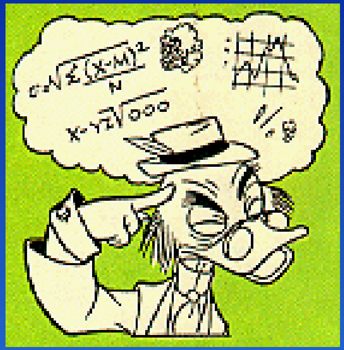
Parameter `*a*` is varied from 0.1 to 2.9 while `*b*` held fixed at *b*=1/3.

Parameter `*b*` is varied from to 0.1 to 0.5 while `*a*` held fixed at *a*=3/4.

Equations of motion from cell to cell of accelerator:

τ_0=reference cell-transit duration, $\tau_s = 2\pi h/\omega$

$T_n = t_n - n\tau_s$ is relative time coordinate



$$E_{n+1} = E_n + eV\cos(\omega T_n)$$

$$T_{n+1} = T_n + \Delta T(E_{n+1}) + (\tau_0 - \tau_s)$$

Conventional case: $\omega = \omega(E)$, ΔT is linear, $\tau_0 = \tau_s$, yields synchronous acceleration: the location of the reference particle is locked to the waveform, or moves adiabatically. Other particles perform (usually nonlinear) oscillations about the reference particle.

Scaling FFAG case: ω fixed, ΔT is nonlinear, yields asynchronous acceleration: the reference particle performs a nonlinear oscillation about the crest of the waveform; and other particle move convectively about the reference.Two possible operation modes are normal $\tau_0 = \tau_s$ and slip $\tau_0 \neq \tau_s$ (see later).

Hamiltonian: $H(x,y,a)=y^3/3 - y - a\sin(x)$

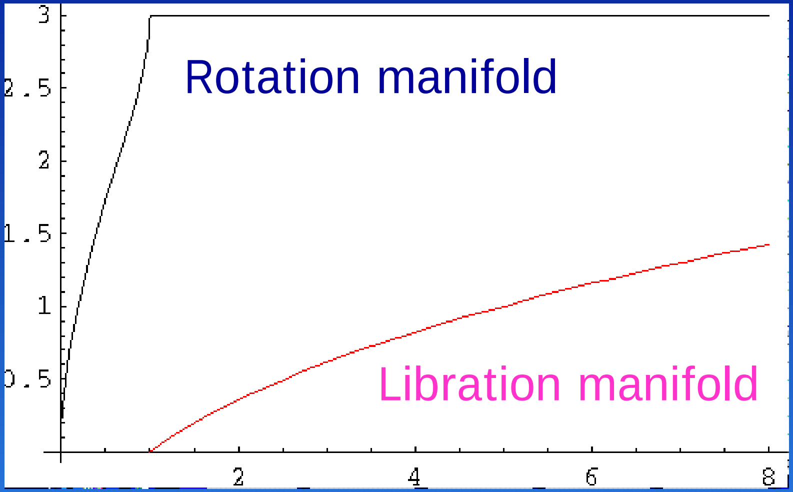
For each value of x, there are 3 values of y: $y_1>y_2>y_3$

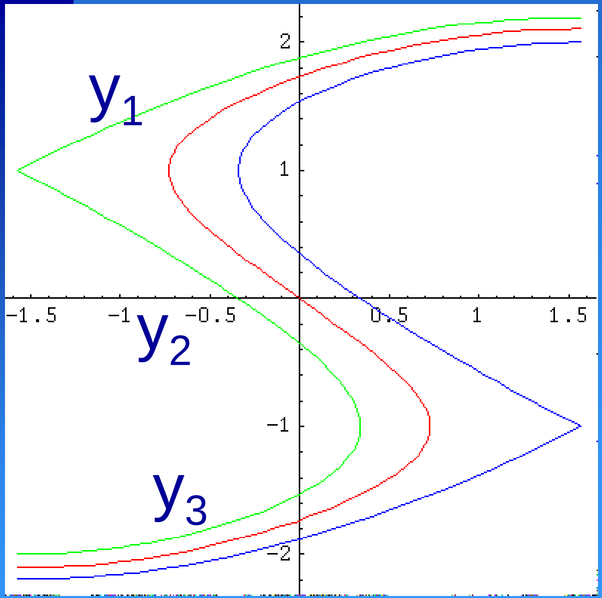
We may write values as $y(z(x))$ where $2\sin(z)=3(b+a\,\mathrm{Sin}x)$

$y_1=+2\cos[(z-\pi/2)/3]$,

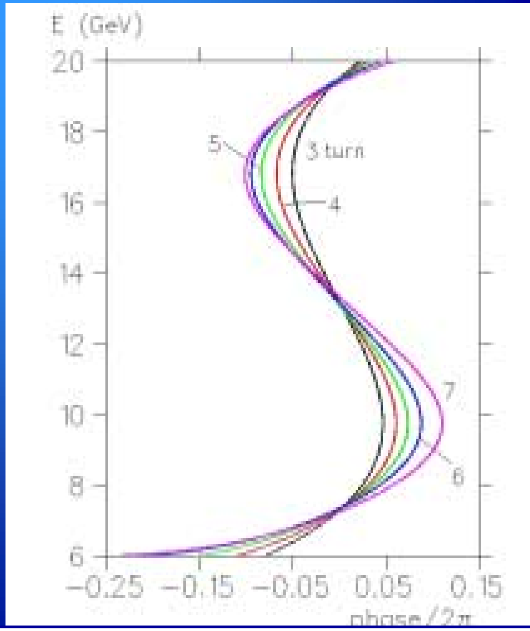
$y_2=-2\sin(z/3)$,

$y_3=-2\cos[(z+\pi/2)/3]$.

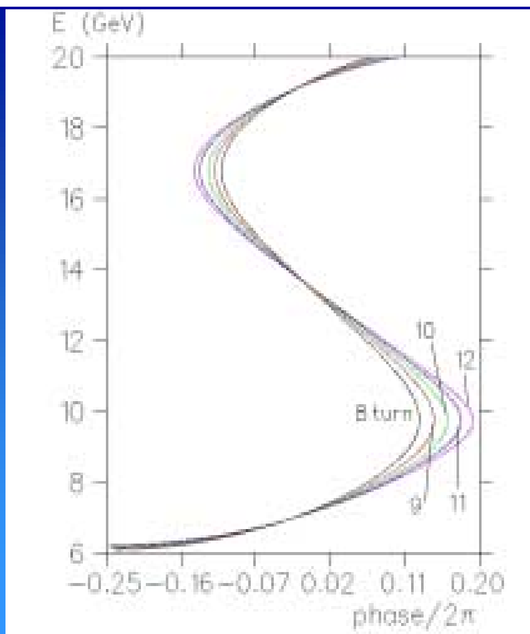


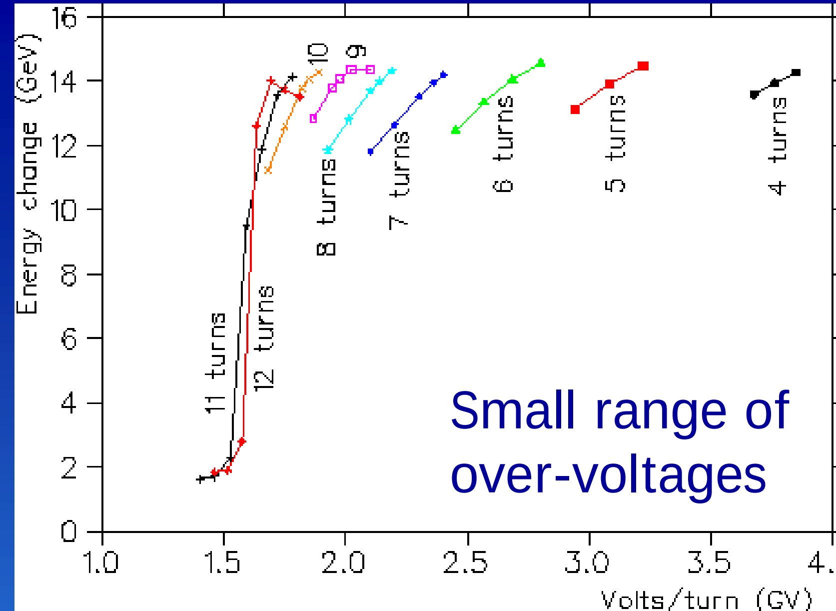


The 3 libration manifolds are sandwiched between the rotation manifolds (or *vice versa*) and become connected when $a \geq 2/3$. Thus energy range and acceptance change abruptly at the critical value.

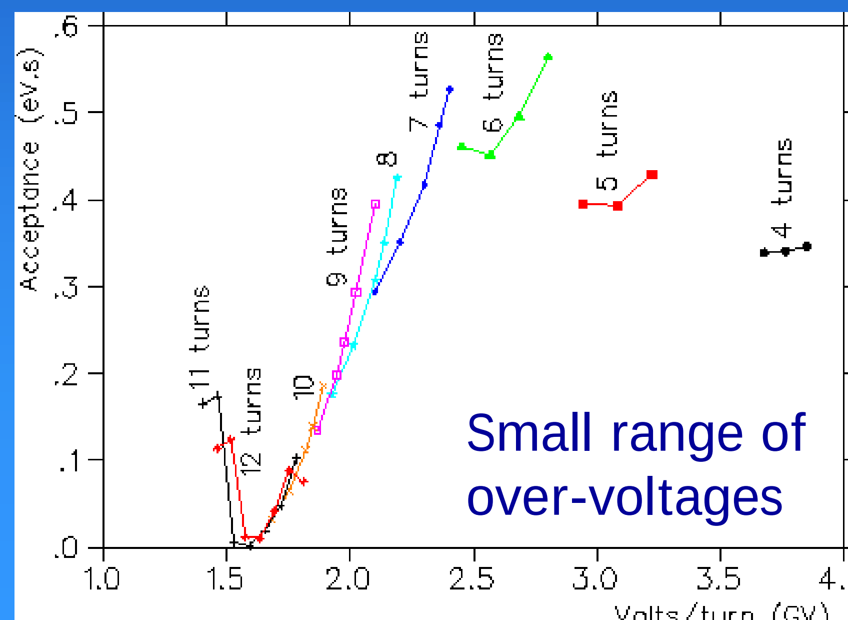


Phase portraits for 3 through 12 turn acceleration; normal rf



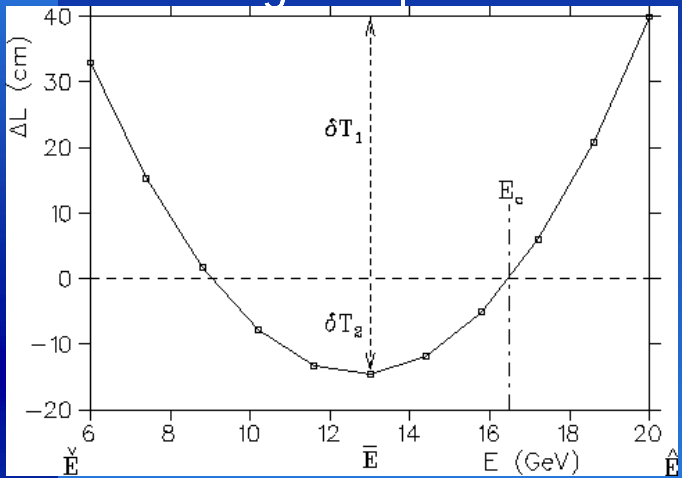


Acceptance and energy range versus voltage for acceleration completed in 4 through 12 turns



$$\sigma = \sqrt{\frac{\delta T_1 + \delta T_2}{\delta T_2}} \qquad \rho = \frac{2}{\omega \delta T_2} \frac{\delta E}{\Delta E}$$

Path length dependence

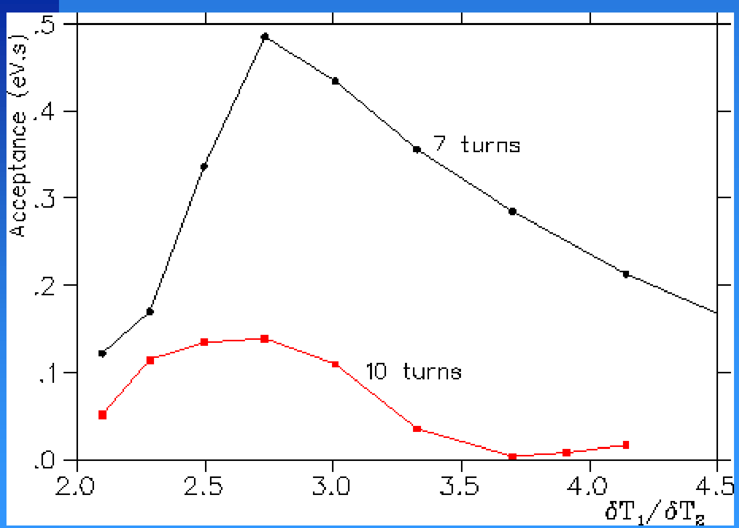


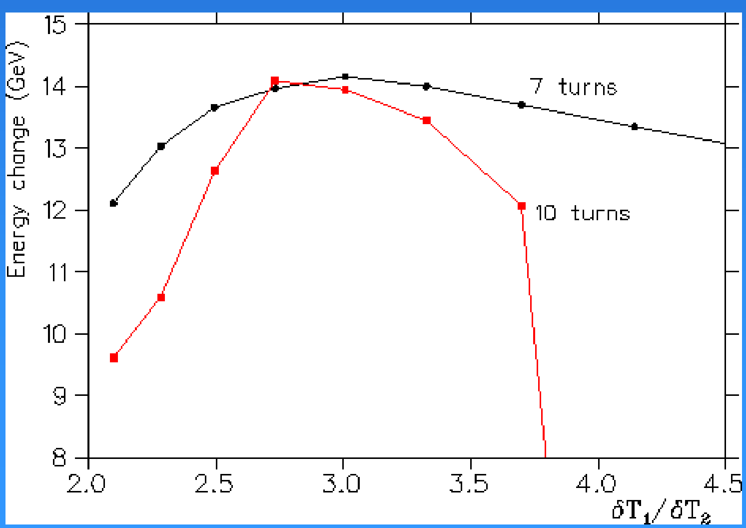
Requirements for lattice

The need to match the path-length parabola to the gutter entrance/exit and fixed points of the phase space has implications for δT_1 & δT_2 etc. Hamiltonian parameter a= $\sigma\times\rho$.

Example: a=2/3 means σ=2, ρ=1/3 and $\delta T_1/\ \delta T_2$=3.

If requirements are violated, then acceptance and acceleration range may deteriorate.

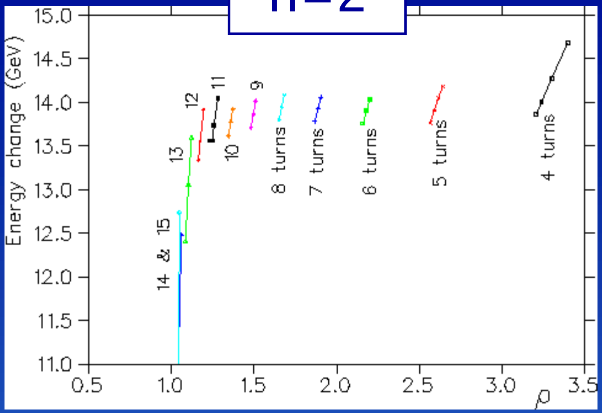


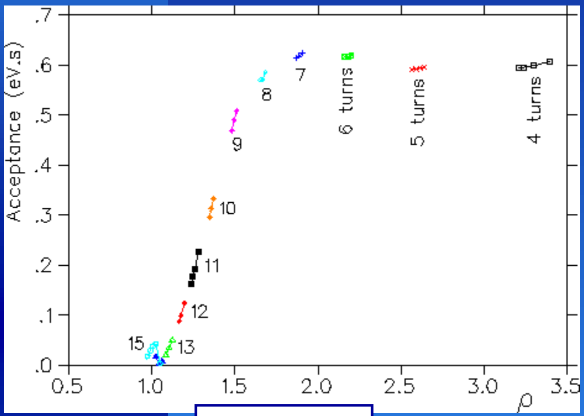


Addition of higher harmonics

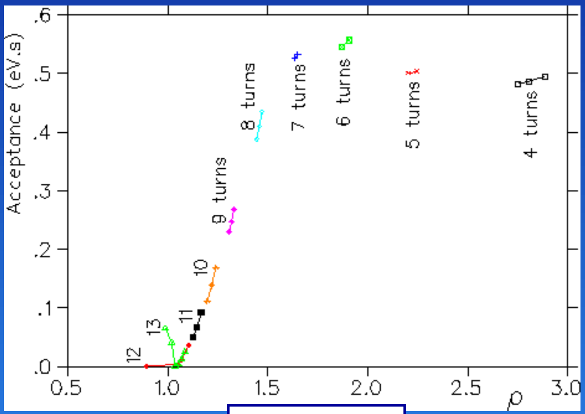
The waveform may be flattened in the vicinity x=0 by addition of extra Fourier components n≥2

n=2



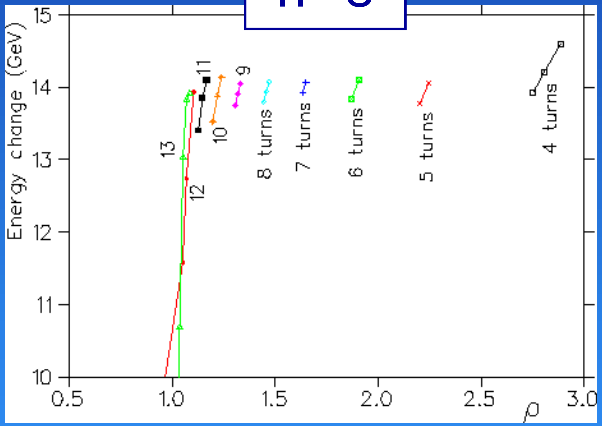


n=2



n=3

n=3



n	Fixed points/(π/2)	a_c	$a_c \times$ boost
1	±1	2/3≈0.6666	0.6666
2	±1.1443	0.48583	0.6478
3	±1	4/7≈ 0.5714	0.6428
4	±0.9614	0.6238	0.6654
5	±1	20/31≈0.645	0.6720

Restoring force $a \times \sin(x) \Rightarrow$

$a[n^3 \sin x - \sin(nx)]/(n^3-n)$

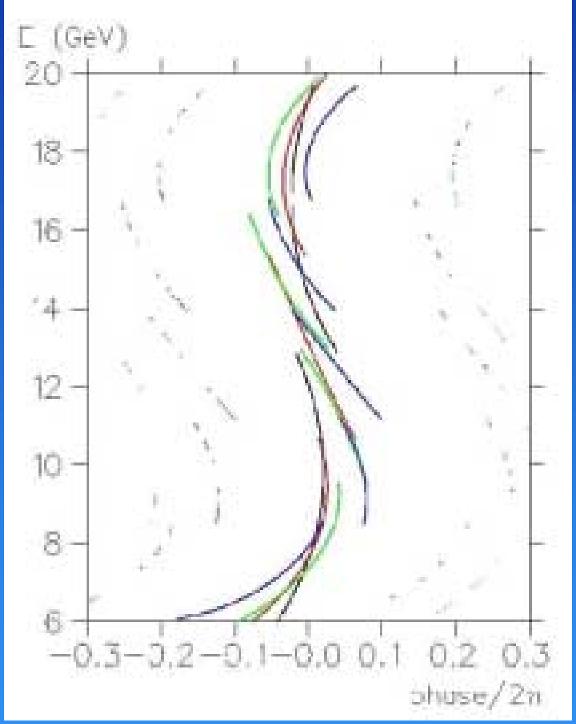
Analogous discontinuous behaviour of phase space, but with revised critical values a_c. Write $\rho = a / a_c$

Asynchronous acceleration with slip rf $\tau_0 \neq \tau_s$

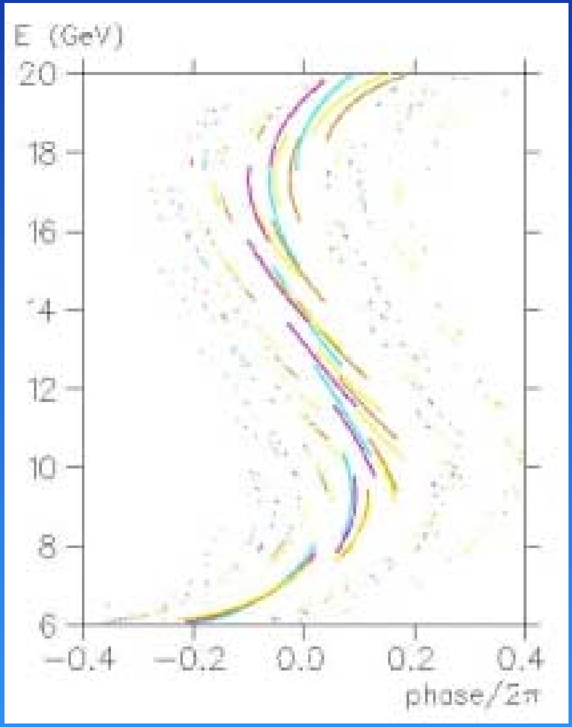
$T_{n+1} = T_n + \Delta T(E_{n+1}) + (\tau_0 - \tau_s)$

And vary initial cavity phases

There is a turn-to-turn phase jumping that leads to a staggering of the phase traces and a smaller r.m.s. variation of rf phase.



Phase portrait: 3-5 turn acceln.



Phase portrait: 6-9 turn acceln.

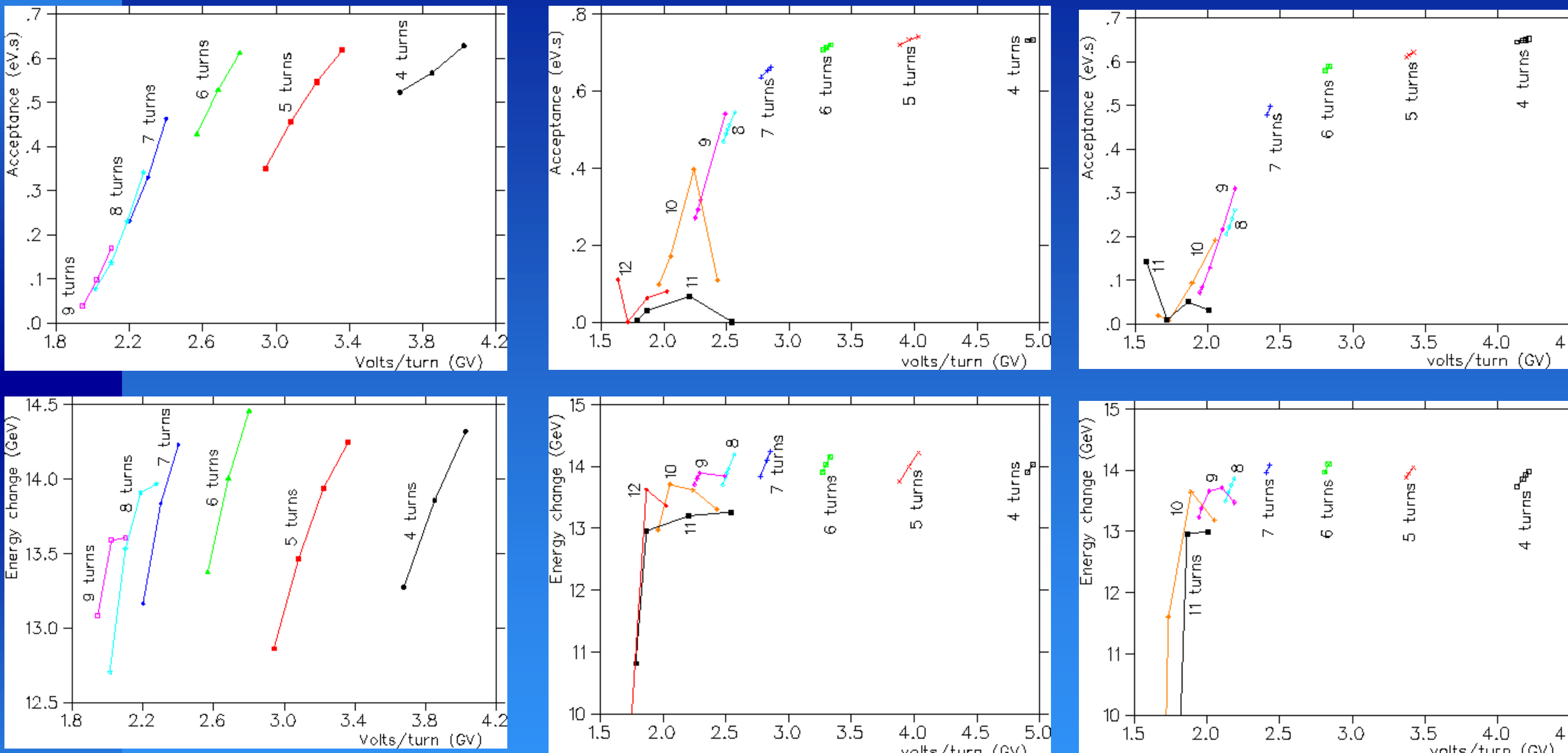
As the number of turns increases, so does the range and number of the outliers. When a significant portion reach `trough phases' the beam fails to receive sufficient average acceleration – limit about 8 turns f.p.c.

Asynchronous acceleration with slip rf and harmonics

fundamental

& 2nd harmonic

& 3rd harmonic



$a_c \approx 0.68$

$a_c \approx 0.55$

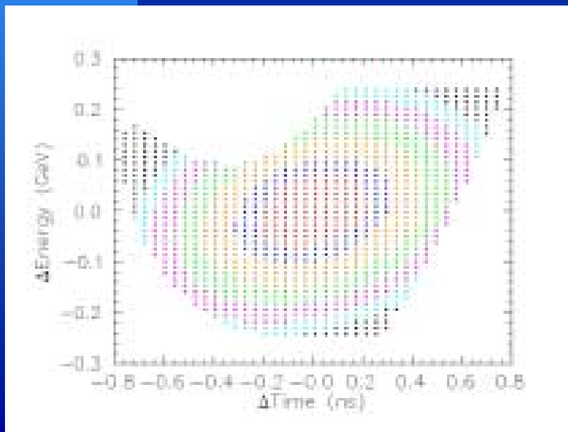
$a_c \approx 0.65$

Acceptance and acceleration range vary abruptly with volts/turn.

Critical a_c can be estimated by tracking of particle ensemble

Phase spaces for normal rf with harmonics

6-turn
fundamental



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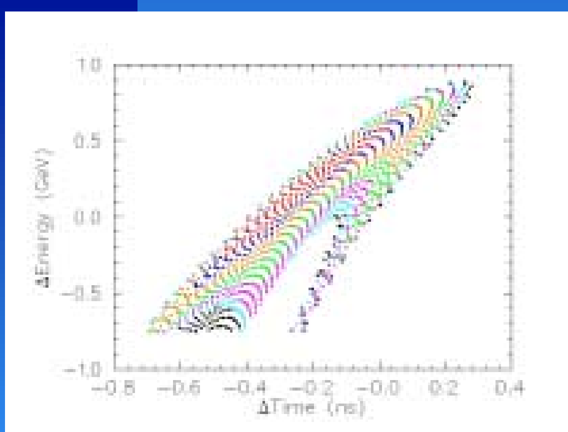
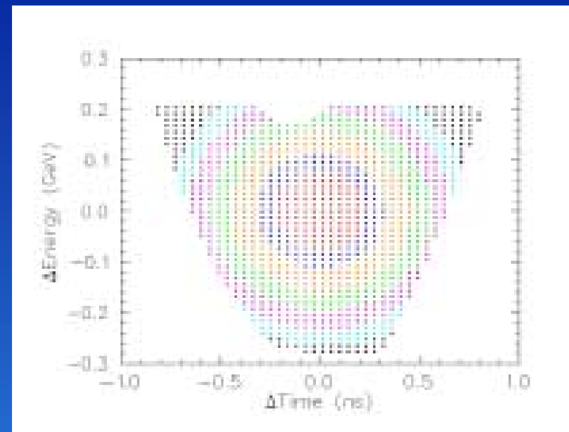


output

Acceptance, ε=.49 eV.s

7-turn
+3rd harmonic



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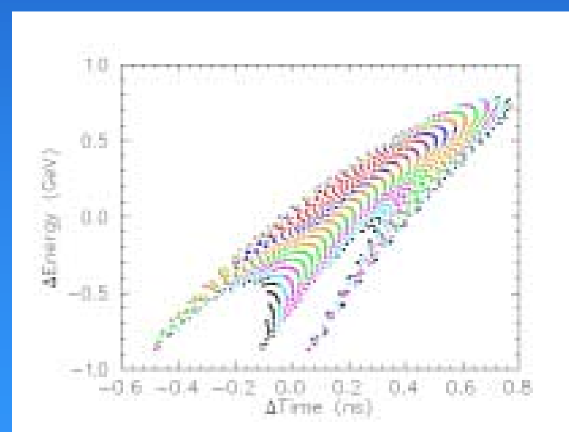
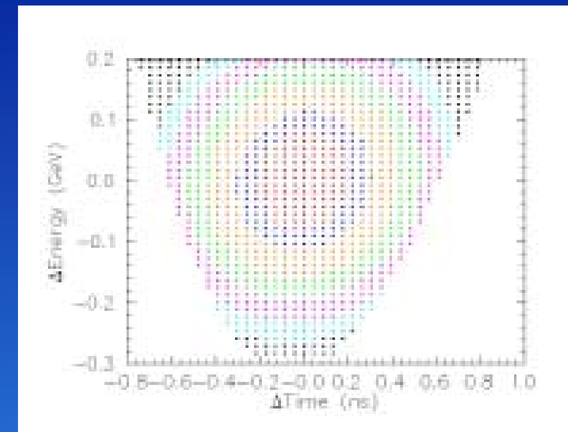


output

Acceptance, ε=.52 eV.s

8-turn
+2nd harmonic



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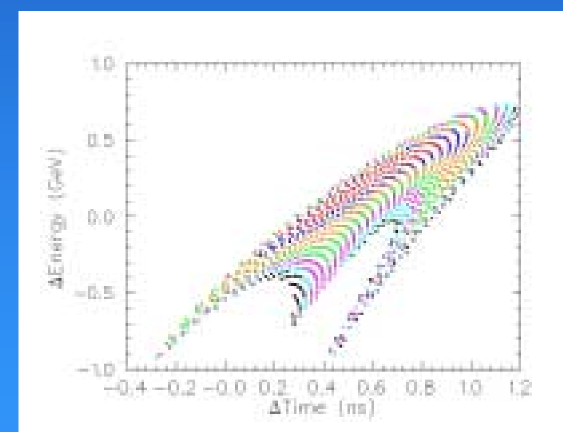
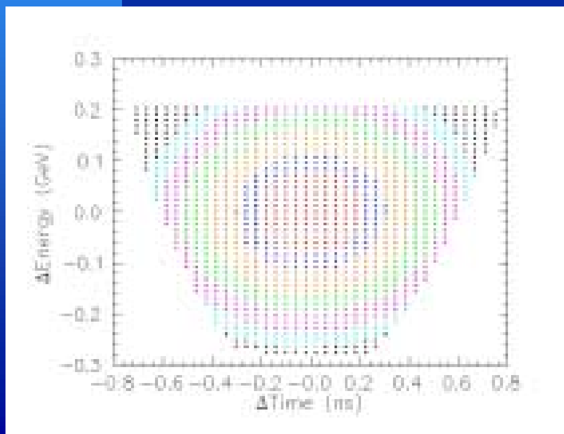


output

Acceptance, ε=.57 eV.s

Phase spaces for slip rf with harmonics

6-turn
fundamental



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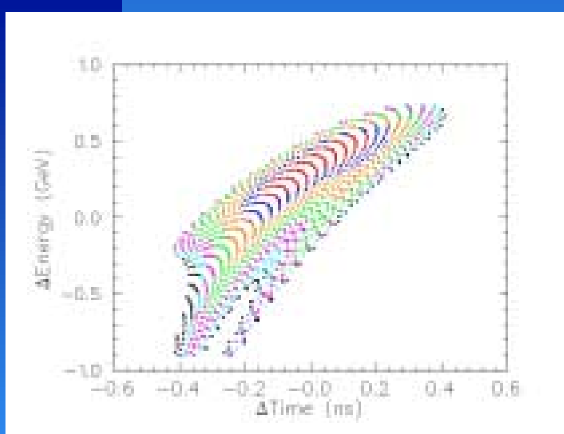
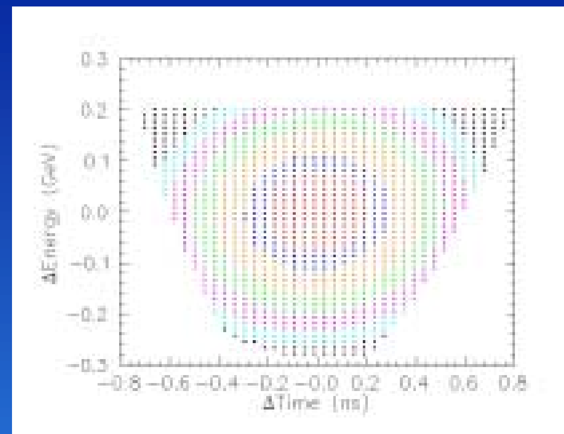


output

Acceptance, ε=.52 eV.s

7-turn
+3rd harmonic



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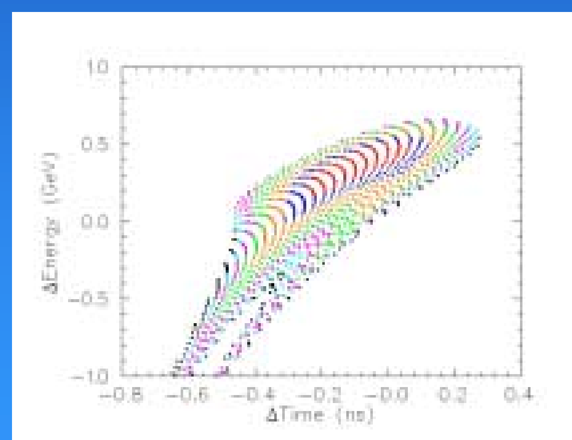
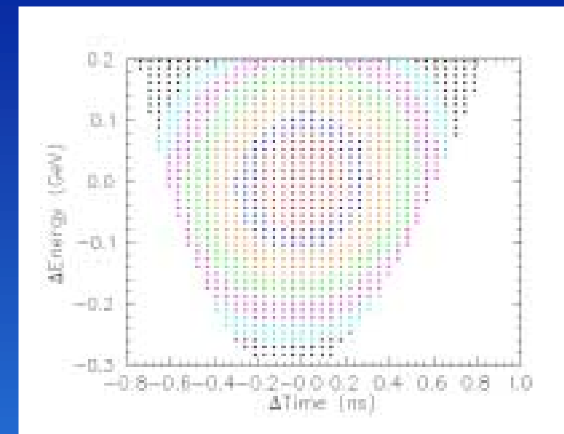


output

Acceptance, ε=.49 eV.s

8-turn
+2nd harmonic



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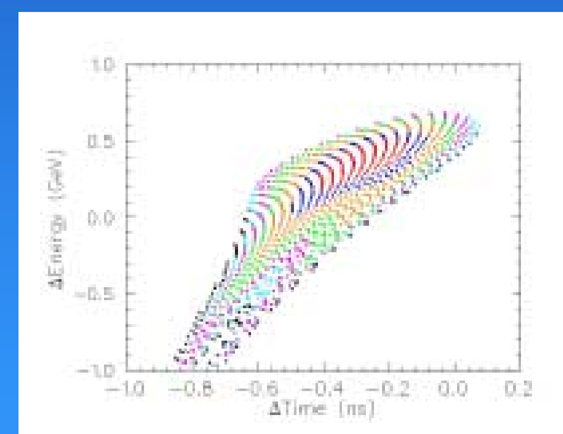


output

Acceptance, ε=.51 eV.s

Conclusions at the time of:

"Methodical, Insidious Progress on Linear Non-scaling FFAGs using High-frequency (≥100 MHz) RF"
C. Johnstone *et al*

- NuFact03
- Columbia Univ., NY
- June 10, 2003

- Performace of nonlinear systems, such as quadratic, cubic, quartic pendula, may be understood in terms of libration versus rotation manifolds and criteria for connection of fixed points.
- Same criteria when higher harmonics added; critical ***a*** renormalized.
- Normal and slip rf operations produce comparable performance. Acceleration is asynchronous and cross-crest.
- The acceptance & acceleration range of both normal and slip rf have fundamental limitations w.r.t. number of turns and energy increment – because gutter paths becomes cut off.
- Regime in which slip rf performs best is one in which energy-increment parameter $\boldsymbol{a} \approx \mathbf{1}$; but this also regime in which phase-space paths become more vertical for normal rf operation.
- Nonlinear acceleration is viable and will have successful application to rapid acceleration in non-scaling FFAGs.

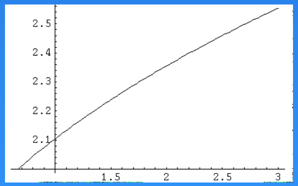
What would we do different? (October 2003)

- Consider different magnet lattices – quadratic dispersion of path length versus momentum may be smaller.

- Reference trajectory for construction of lattice and transverse dynamics is not necessarily the same as that for longitudinal dynamics. This allows zeros of path length and/or $\delta T_1/\delta T_2$ to be adjusted.

- Pay more careful attention of matching longitudinal phase space topology to desired input/output particle beam. Perhaps adjust $\delta T_1/\delta T_2$ as function of ***a***.

- Match orientation and size of beam to the libration manifold. For example, if inject/extract at $x = \pm\pi/2$, $\sigma = 2\cosh[(1/3)\mathrm{arccosh}(a/a_c)]$, $\rho = -1 + \sigma^2/3$



- If F0D0 or regular triplet, dispersion of arrival times (at extraction) is a symmetric minimum about central trajectory H(x,y,a)=0; inject beam on to that trajectory.