Scaling FFAGs and the "Smooth Approximation"

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There has been a lot of discussion as to how increasing the field index, ***k,*** will help many problems with an FFAG. Here I wish to use the "smooth approximation" to illustrate the effect of increasing ***k*** on a number of important properties. Most expressions quoted here come from significant papers published in the 1950s. From these, we will see that increasing ***k*** has a number of significant down sides.

For moderate betatron phase advances per sector (not over 90°), simple expressions derived from the "smooth approximation" yield tunes that are accurate to around 10% of the rms azimuthal ripple. These expressions are very helpful in determining the behavior of important accelerator properties on its parameters. The derivation is found in Appendix A in the 1956 Physical Review paper by Simon, Kerst, Jones, Laslett, and Terwilliger, *Fixed-Field Alternating-Gradient Particle Accelerators* (p. 1937). In the following, I use the expressions found in the paper by Frank Cole, *Typical Designs of High Energy FFAG Accelerators*, in the 1959 Conference on High-Energy Accelerators and Instrumentation, CERN (1959) p. 82.

Derivation [from Cole]

We write the most general magnetic field in the median plane for a scaling machine as:

$$\left\{ \begin{aligned} B_r &= B_\theta = 0 \\ B_z &= B_0 \left(\frac{r}{r_0} \right)^k f(\psi) \\ &\textit{where} \\ f(\psi) &= \sum_{n=0}^{\infty} (g_n \cos n\psi + f_n \sin n\psi); \quad g_0 = 1 \\ \psi &= K \ln \frac{r}{r_0} - N\theta = N \left(\tan\zeta \, \ln \frac{r}{r_0} - \theta \right) \\ &\textit{for} \\ K &= N \tan\zeta \end{aligned} \right.$$

Define the *flutter*, F, and the associated nameless function, G:

$$\left\{ \begin{aligned} F^2 &= \sum_{n=1}^{\infty} (g_n^2 + f_n^2); \quad F^2 = 2\left[\frac{\langle B^2\rangle - \langle B\rangle^2}{\langle B\rangle^2} \right] \\ G^2 &= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{g_n^2 + f_n^2}{n^2 N^2} \right) \end{aligned} \right.$$

Note that G is related to the azimuthal integral of $f(\psi)$ the way F is related to $f(\psi)$. The smooth approximation yields the following formulae for the tunes:

$$\begin{cases} \nu_x^2 = k + 1 + k^2 G^2 \\ \nu_y^2 = -k + k^2 G^2 + F^2 \left(\dfrac{K^2}{N^2} + \dfrac{1}{2} \right) \end{cases}$$

Here the terms in k alone are the focusing terms due to the average magnetic gradient, k^2G^2 terms show the focusing contribution of the alternate gradient focusing, $\frac{1}{2}F^2$ is the Thomas focusing term, and the term containing K is the focusing due to the spiral angle. To increase the vertical tune, we need to add higher and higher harmonics, moving from a sine wave azimuthal dependence towards a square wave. (Warning: in some other works, the flutter function, $\mathbf{F}^2$, is sometimes designated, $\mathbf{F}$, and sometimes is defined as including a factor of ½.). We use Cole's definition.

Tune Discussion

Increasing k clearly increases the radial tune, which we wish to keep less than $4N$. However, it decreases the vertical tune, making it difficult to keep vertical motion stable without adding a spiral angle [$K = N \tan \zeta$]. Consider two extreme cases: the first where the azimuthal profile is simply a sine wave, and the second where the azimuthal profile is an antisymmetric square wave. We expect that most realized azimuthal profiles will lie between these two limits and thus the derived tunes should also lie between those calculated at these limits.

Sine Wave Azimuthal Dependence

Let $f(N\theta) = 1 + a\cos(N\theta)$ then $F = a$; $G = \dfrac{1}{\sqrt{2}}\dfrac{a}{N}$ from which we find

$$\nu_x^2 = k + 1 + \frac{1}{2}\frac{k^2 a^2}{N^2}$$

$$\nu_y^2 = \frac{1}{2}a^2 - k + \frac{1}{2}\frac{k^2 a^2}{N^2} = \frac{1}{2}\frac{a^2}{N^2}\left[k - \frac{N^2}{a^2}\right]^2 + \frac{1}{2}a^2 - \frac{1}{2}\frac{N^2}{a^2}$$

a is the amplitude of the azimuthal oscillation of the field strength. These are the approximate lower bounds on the square of the tunes. The minimum ν_y^2 occurs for $k = N^2/a^2$; here we would need $N < a^2$ for the vertical motion to be stable. Keeping the vertical motion stable in the absence of a spiral angle becomes increasingly difficult as the number of sectors is increased for this low flutter example.

Square Wave Azimuthal Dependence

Let $$\begin{cases} f(N\theta) = 1 + a & 0 < N\theta < \pi \\ f(N\theta) = 1 - a & \pi < N\theta < 2\pi \end{cases}$$

We find $$\begin{cases} f_0 = 1 & f_1 = \dfrac{4a}{\pi} \\ f_2 = 0 & f_3 = \dfrac{4a}{3\pi} \\ f_4 = 0 & f_5 = \dfrac{4a}{5\pi} \quad .... \end{cases}$$

Using expression 48.4 in Dwight's tables, we find $F^2 = 2a^2$, which is double the value for the sin wave.

The expression for G^2 is a little more complicated, but Dwight's 47.3 puts the sum in terms of the Bernoulli number $B_2 = \frac{1}{30}$, thus $G^2 = \frac{\pi^2 a^2}{12N^2}$, which is 64½ % larger than for the sin wave..

The approximate squares of the tunes are $$\begin{cases} \nu_x^2 = k + 1 + \dfrac{k^2 \pi^2 a^2}{12N^2} \\ \nu_y^2 = a^2 - k + \dfrac{k^2 \pi^2 a^2}{12N^2} \end{cases}.$$

These two represent reasonable limits on the flutter obtainable. Thus we can ask how large can $\boldsymbol{k}$ be and still have stable vertical motion. For $0<a<1$, the magnetic field is always positive. From the above, we see there are solutions for a radial FFAG with no negative fields. A reasonable radial ffag would have a~3 (the positive field is twice as strong as the negative field).

Symmetric Lattice

Assuming that the injected beam has similar emittances in the two planes, it is desirable that the radial and vertical tunes be about the same. This condition yields: $2k = -1 + F^2 \left(\frac{K^2}{N^2} + \frac{1}{2} \right)$.

For a radial machine, using the above substitutions, we find

$$2k = \begin{cases} \dfrac{1}{2}a^2 - 1 & \textit{for } \sin \textit{ wave} \\ a^2 - 1 & \textit{for square wave} \end{cases}.$$

These represent lower and upper bounds for most any azimuthal profile that might be considered. This is a significant restriction on $\boldsymbol{k}$. For example, for a magnet with a negative field that is one half the magnitude of the positive field, we have $\boldsymbol{a=3}$ and $\frac{7}{4} \le k \le 4$. For higher values, we must add a spiral to the magnet or else drop the condition that $\nu_y \sim \nu_x$. For high energy machines, we generally will need $\nu_y \ll \nu_x$.

Average Radius

Using Gode Wüstefeld's expressions (see appendix) for the square wave example, we find $R = \frac{(B\rho)}{B_0}(1+a)$. So increasing $\boldsymbol{a}$ increases the machine size and expense.

Other Properties

Momentum Compaction: $\frac{dp}{p} = (k+1)\frac{dr}{r}$ so increasing $\boldsymbol{k}$ increases the compaction.

Acceptance: The acceptance in the radial plane is usually limited by the $3\nu_x = N$ resonance. Cole states that the maximum stable radial amplitude is given by (using our notation) :

$A = \frac{8}{3}\frac{2N}{ak^2}\left|\nu_x - \frac{1}{3}N\right|$ for a radial machine when close to the resonance.

Thus the acceptance is inversely proportional to the 4th power of $\boldsymbol{k}$ and the square of the field oscillation amplitude.. Increasing $\boldsymbol{k}$ very rapidly reduces the radial acceptance.

Magnet Size: Cole shows that the total magnet volume is approximately proportional to $\frac{r_{\max}^3}{k^2}$ so large k and small r_{max} will reduce costs. Thus one needs to find an optimum balance for the field exponent to provide an adequate acceptance while keeping the magnet size constrained.

Appendix

The Geometric Size of the FFAG

Gode Wüstefeld, BESSY {PAC 1985 and PAC 1987}

I reprint the information from these two PAC papers because this is really good demonstration as to the power of the smooth approximation. It is also found in two internal reports: ANL ASPUN 10 and KFA ABT Note 28/85.

Consider a lattice of N cells containing one positive magnet of length P/N with field B_0 and one negative magnet of length M/N and field $-mB_0$. Thus P is the total length of all positive magnets, and M is the total length of all negative magnets. The distribution of the magnets within the cell is completely arbitrary (as long as they are not superimposed!) The following applies to a doublet cell and to triplet cells (where one or the other magnet type is split into two parts).

The average field around the ring is given by

$<B> = \dfrac{B_0\,(P-mM)}{2\pi R}$, where R is the average radius of the machine.

The magnetic rigidity is: $B\rho = <B>R = B_0\,\dfrac{P-mM}{2\pi R}$.

We can also easy get the mean square field: $<B^2> = B_0^2\,\dfrac{(P+m^2M)}{2\pi R}$.

Introduce $\alpha = mM/P$; then the flutter can be written as

$\tfrac{1}{2}F^2+1 = \dfrac{<B^2>}{<B>^2} = \dfrac{B_0 R}{B\rho}\,\dfrac{1+m\alpha}{1-\alpha}$, from which we can solve for the average radius R:

$$R = (\tfrac{1}{2}F^2+1)\,\frac{B\rho}{B}\,\frac{1-\alpha}{1+m\alpha} = \frac{B\rho}{B}\,\frac{1-\alpha}{1+m\alpha}\left[\frac{\nu_x^2+\nu_y^2-2k^2G^2-1}{1+\tan^2\zeta}+1\right].$$

$$P+m^2M = \frac{2\pi}{R}\left(\frac{B\rho}{B_0}\right)^2\left[\frac{\nu_x^2+\nu_y^2-2k^2G^2-1}{1+\tan^2\zeta}+1\right];$$

this is closely related to the total magnet length in the ring (the same when ***m=1***). These formulae are accurate to the extent the smooth approximation is accurate. We immediately see the advantage of strong magnets, as doubling the magnetic field reduces the total azimuthal magnet length (at least where ***m=1***) by 75%.