Solenoidal Channel Linear Dynamics

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Motivation:

Solenoidal lossless transport

Solenoidal channels matching

Linear theory of solenoidal channels

Analytical solution of the single particle dynamics

Envelopes solutions

Examples of applications and outlook

G.Franchett

First order equation of motion in a solenoidal channel

$$\boxed{\begin{aligned} x'' - Sy' - \tfrac{1}{2}S'y &= 0 \\ y'' + Sx' + \tfrac{1}{2}S'x &= 0 \end{aligned}}$$

where $S = \frac{qB_s(s)}{p}$ (units mks).

q [C] is charge of the particle,

B_s [T] is the longitudinal magnetic field,

p [Kg m/s] is the longitudinal momentum.

Complex notation

Defining $z = x + i\, y$ $\Rightarrow$ $z'' + iSz' + \frac{i}{2}S'z = 0$

In the Larmor Frame

Defining the new complex variable $w = ze^{i\phi}$ $\Rightarrow$ $w'' + \phi'^2 w = 0$

$$\phi(s) = \frac{1}{2}\int_0^s S(t)dt$$

$$w_r'' + \phi'^2 w_r = 0 \quad \Rightarrow \quad \begin{cases} w_r'' + \phi'^2 w_r = 0 \\ w_i'' + \phi'^2 w_i = 0 \end{cases}$$

At each of the two component one can apply the Courant Snyder theory

Transport in (w_r, w_r') and (w_i, w_i') is well defined in terms of solenoidal channels Twiss parameters defined as

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \frac{S^2}{4}\beta^2 = 1, \qquad \alpha = -\frac{\beta'}{2}, \qquad \psi = \int_0^s \frac{1}{\beta(t)}dt$$

Solution of the equation of motion in the laboratory frame

$$
\begin{aligned}
x &= \sqrt{\beta\epsilon_1}\sin(\psi+\delta_1)\cos\phi + \sqrt{\beta\epsilon_2}\sin(\psi+\delta_2)\sin\phi \\
y &= \sqrt{\beta\epsilon_2}\sin(\psi+\delta_2)\cos\phi - \sqrt{\beta\epsilon_1}\sin(\psi+\delta_1)\sin\phi \\
x' &= \sqrt{\frac{\epsilon_1}{\beta}}[\cos(\psi+\delta_1)\cos\phi - \alpha\sin(\psi+\delta_1)\cos\phi - \beta\phi'\sin(\psi+\delta_1)\sin\phi]+ \\
&+ \sqrt{\frac{\epsilon_2}{\beta}}[\cos(\psi+\delta_2)\sin\phi - \alpha\sin(\psi+\delta_2)\sin\phi + \beta\phi'\sin(\psi+\delta_2)\cos\phi] \\
y' &= \sqrt{\frac{\epsilon_1}{\beta}}[-\cos(\psi+\delta_1)\sin\phi + \alpha\sin(\psi+\delta_1)\sin\phi - \beta\phi'\sin(\psi+\delta_1)\cos\phi]- \\
&+ \sqrt{\frac{\epsilon_2}{\beta}}[\cos(\psi+\delta_2)\cos\phi - \alpha\sin(\psi+\delta_2)\cos\phi - \beta\phi'\sin(\psi+\delta_2)\sin\phi]
\end{aligned}
$$

$$\psi = \text{betatron phase advance}, \ \phi = \text{Larmor phase advance}$$

$$\delta_1, \delta_2 = \text{initial particle phases}$$

Particle emittances ➡

$$
\begin{aligned}
\epsilon_1 &= \epsilon_{x0} + \phi'^2\beta y_0^2 - 2\alpha\phi' x_0 y_0 - 2\beta\phi' y_0 x_0' \\
\epsilon_2 &= \epsilon_{y0} + \phi'^2\beta x_0^2 + 2\alpha\phi' x_0 y_0 + 2\beta\phi' x_0 y_0'
\end{aligned}
$$

$$\epsilon_{x0}, \epsilon_{y0} = \text{usual Courant Snyder invariants}$$

Application: lossless beam transport

$$r^2 = x^2 + y^2 = \beta(\epsilon_1 \sin^2(\psi + \delta_1) + \epsilon_2 \sin^2(\psi + \delta_2))$$

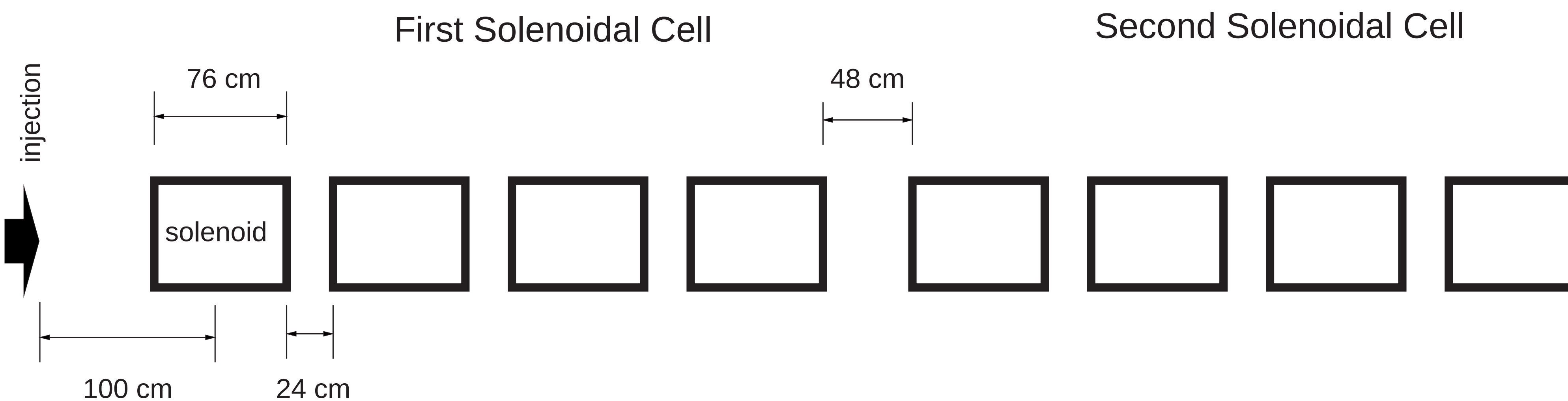
Lossless condition $$\beta(\epsilon_1 \sin^2(\psi + \delta_1) + \epsilon_2 \sin^2(\psi + \delta_2)) < R^2$$

R = radius of the beam pipe

Conservative condition (but easier) $$\epsilon_1 + \epsilon_2 \le \frac{R^2}{\beta_{max}}$$

$$\begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \end{pmatrix} \xrightarrow{\text{by using eq. of } \beta} \begin{pmatrix} \beta_{max} \\ \beta_0 \\ \alpha_0 \end{pmatrix} \xrightarrow{\text{by using eqs. of } \epsilon_1, \epsilon_2} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \longrightarrow \sqrt{\beta_{max}(\epsilon_1 + \epsilon_2)}$$

Application: two 44 MHz cooling cells



$$B_s(s) = B_0 \frac{\sqrt{(0.5d)^2 + R_s^2}}{d} \left(\frac{0.5d - s}{\sqrt{(0.5d - s)^2 + R_s^2}} + \frac{0.5d + s}{\sqrt{(0.5d + s)^2 + R_s^2}} \right),$$

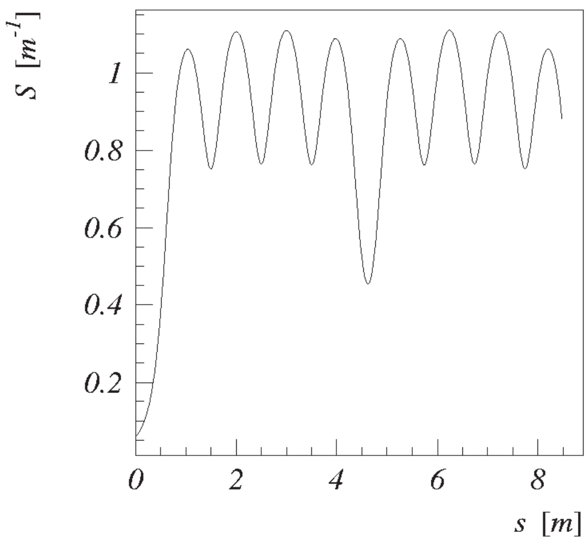
$d = 0.76$ m is the length of the solenoid,

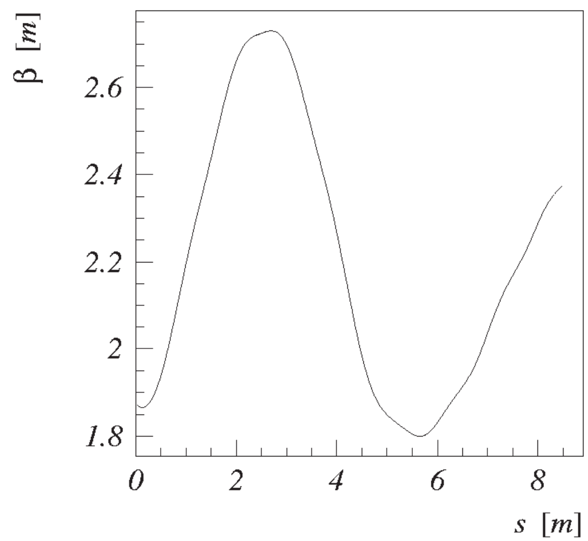
$R_s = 0.3$ m is the solenoid radius,

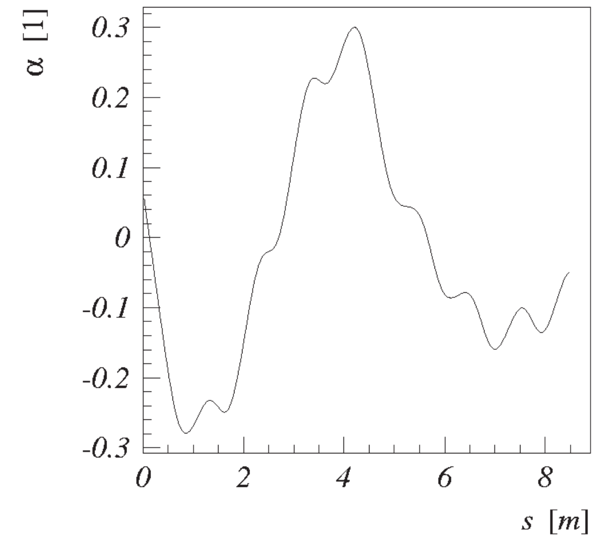
$B_0 = 2$ T is the magnetic field in the center of the solenoid.

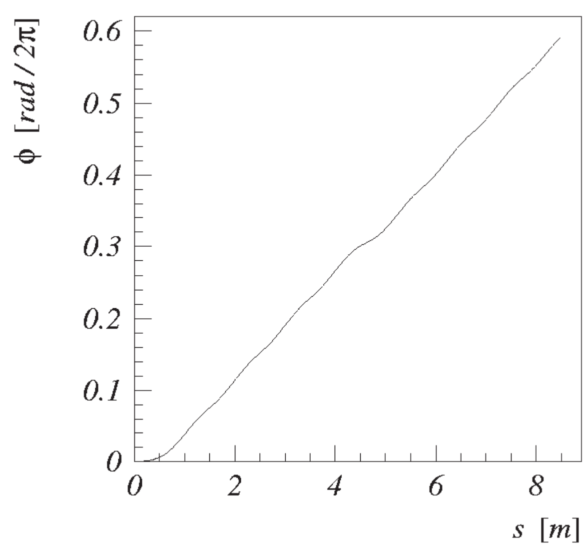
$\epsilon_1 = 1$ cm-rad, and $\epsilon_2 = 2$ cm-rad, 30 particles.

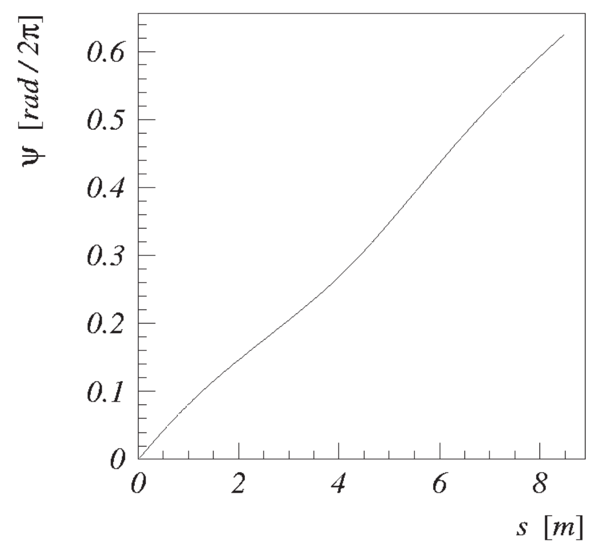
For muons with kinetic energy of $E = 200$ MeV we find that $S = 1$ m^{-1}.


S [m^-1]
1
0.8
0.6
0.4
0.2
0
2
4
6
8
s [m]

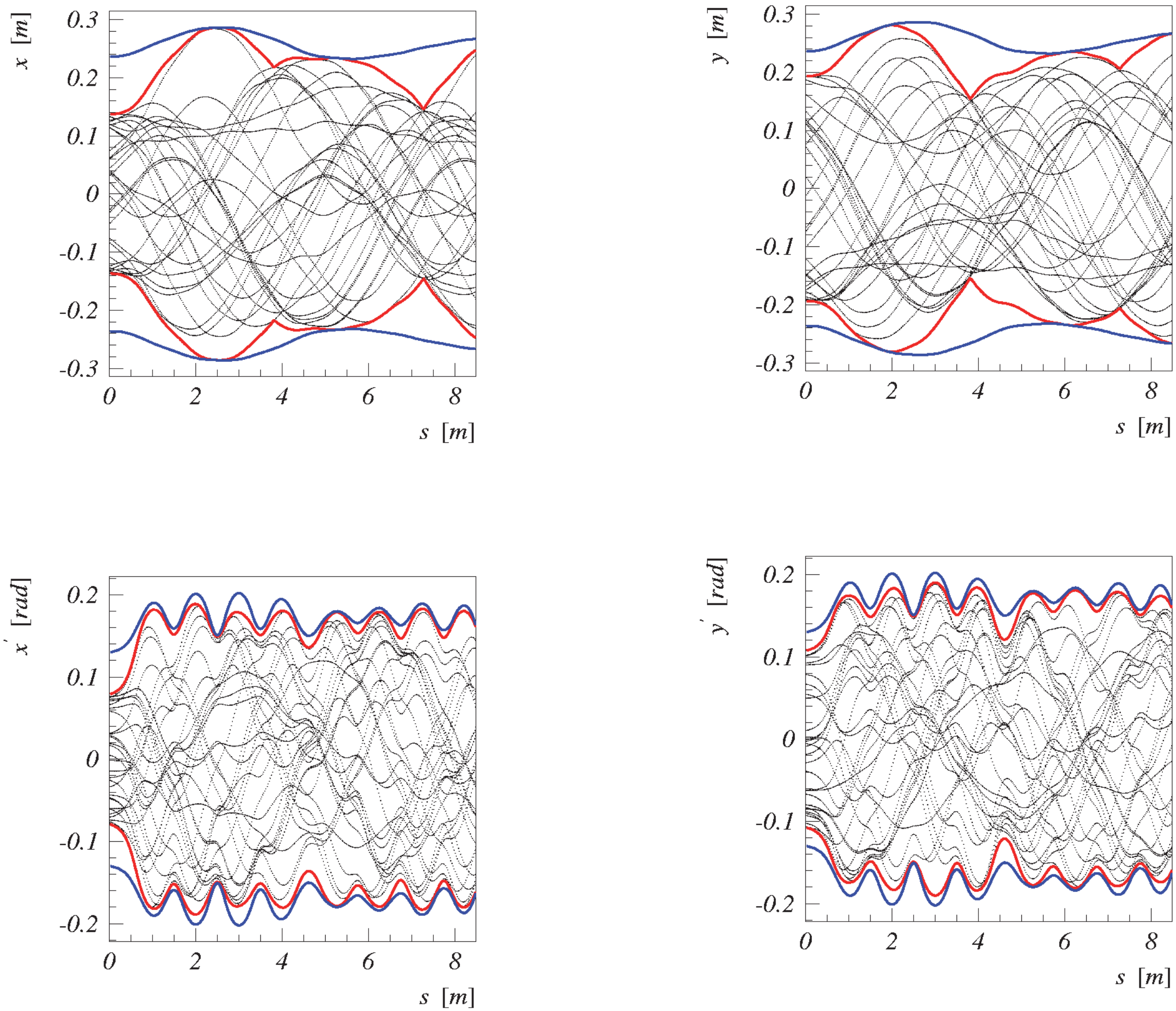

β [m]
2.6
2.4
2.2
2
1.8
0
2
4
6
8
s [m]


α [1]
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0
2
4
6
8
s [m]


φ [rad / 2π]
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
s [m]


ψ [rad / 2π]
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
s [m]

Application: beam envelope



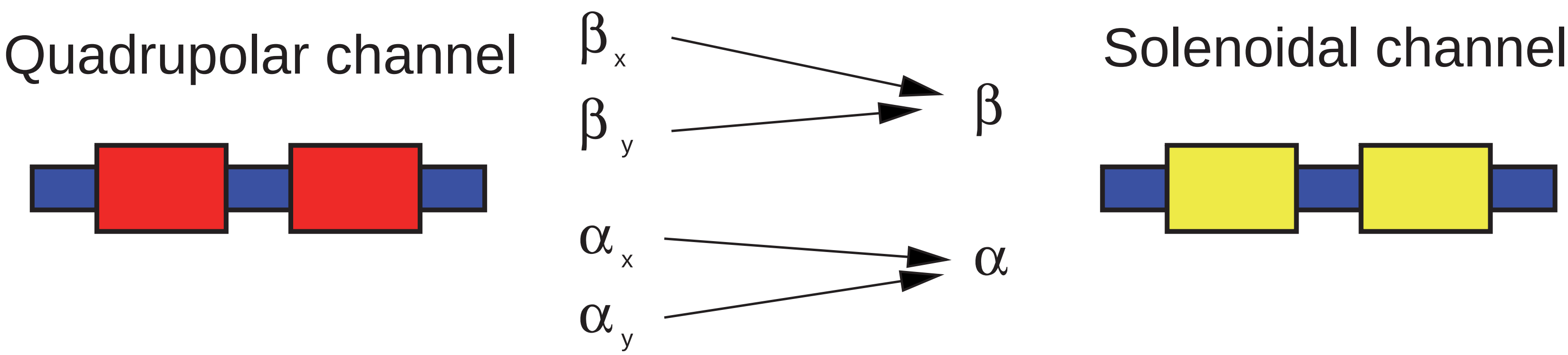
Solenoidal Channels Matching

The matching of the Twiss parameters of the two solenoidal channels does not change the acceptance of each channel.

A mismatch causes a change in the beta of the second channel and consequently the acceptance of the second channel changes

Solenoidal Channels and Quadrupolar Channels Matching

In order to keep the quadrupolar channel single particle emittances a matching section from quadrupolar channel and solenoidal is neede



Conclusion

We presented the solution of the single particle dynamics in a general solenoidal channel in terms of Twiss parameters and invariants

We presented the explicit analytical form of the single particle invariants

Analytical envelope solution can be found

Twiss parameters allow linear design and matching

This theory give simple tools for first design: nonlinear effects on the beam dynamics must be evaluated with a multiparticles tracking code