Balbekov (Tetra) Ring

Simulation Results in COSY

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See: ICAP 2002 Proceedings
Muon Beam Ring Cooler by V. Balbekov
(Pictures: Courtesy of Balbekov)

Figure 1: Layout and parameters of the solenoid based ring cooler.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circumference</td>
<td>36.963 m</td>
</tr>
<tr>
<td>Nominal energy at short SS and bends</td>
<td>250 MeV</td>
</tr>
<tr>
<td>Bending field</td>
<td>1.453 T</td>
</tr>
<tr>
<td>Norm. field gradient</td>
<td>0.5</td>
</tr>
<tr>
<td>Max. solenoid field</td>
<td>5.155 T</td>
</tr>
<tr>
<td>RF frequency</td>
<td>205.69 MHz</td>
</tr>
<tr>
<td>Accelerating gradient</td>
<td>15 MeV/m</td>
</tr>
<tr>
<td>LH$_2$ absorber length</td>
<td>128 cm</td>
</tr>
<tr>
<td>LiH wedge absorber</td>
<td>14 cm</td>
</tr>
<tr>
<td>Grad. of energy loss</td>
<td>0.75 MeV/cm</td>
</tr>
</tbody>
</table>
Map Method

- The transfer map is the flow of the system ODE.
  \[ \vec{z}_f = \mathcal{M}(\vec{z}_i, \delta) \]

- The Differential Algebraic (DA) method allows the computation and manipulation of maps efficiently and elegantly.

- For a repetitive system, only one cell has to be computed. Thus, much faster than tracking codes.

- The Normal Form method can be used for analysis of nonlinear behavior.
The Particle Optical Equations of Motion

\[ x' = a \cdot (1 + hx) \cdot \frac{p_0}{p_z} \]

\[ y' = b \cdot (1 + hx) \cdot \frac{p_0}{p_z} \]

\[ l' = (1 + \delta_m) \cdot (1 + hx) \cdot \frac{1 + \eta}{1 + \eta_0} \cdot \frac{p_0}{p_z} \]

\[ a' = \left( (1 + \delta_m) \cdot \frac{1 + \eta}{1 + \eta_0} \cdot \frac{p_0}{p_z} \cdot \frac{E_x}{\chi E_0} - \frac{B_y}{\chi M_0} + b \cdot \frac{p_0}{p_z} \cdot \frac{B_z}{\chi M_0} \right) \]

\[ b' = \left( (1 + \delta_m) \cdot \frac{1 + \eta}{1 + \eta_0} \cdot \frac{p_0}{p_z} \cdot \frac{E_y}{\chi E_0} + \frac{B_x}{\chi M_0} - a \cdot \frac{p_0}{p_z} \cdot \frac{B_z}{\chi M_0} \right) \]

\[ (1 + hx) \cdot (1 + \delta_z) \]

\[ \chi_{E0} = \frac{p_0 \cdot v_0}{z_0 e}, \quad \chi_{M0} = \frac{p_0}{z_0 e} \]

\[ \eta = \left( \frac{K_0 \cdot (1 + \delta_k) - z_0 \cdot e \cdot (1 + \delta_z) \cdot V(x, y, s)}{m_0 c^2 \cdot (1 + \delta_m)} \right) \]

\[ \frac{p_z}{p_0} = \sqrt{(1 + \delta_m)^2 \cdot \frac{\eta(2 + \eta)}{\eta_0(2 + \eta_0)} - a^2 - b^2} \]

\[ a = \frac{p_x}{p_0} \quad b = \frac{p_y}{p_0} \]
DA Fixed Point Theorem

**Differential Algebra** $\mathcal{D}_v^D$ : in $v$ variables up to order $n$.

**Definition (Depth)** To any element $[f] \in \mathcal{D}_v^D$ we define the depth

$$\lambda([f]) = \begin{cases} 
\text{Order of first nonvanishing derivative of } f & \text{if } [f] \neq 0 \\
n + 1 & \text{if } [f] = 0
\end{cases}.$$

**Definition (DA Contracting Operator)** Let $\mathcal{O}$ be an operator on the set $M \subset \mathcal{D}_v^m$. $\mathcal{O}$ is contracting on $M$ if for any $\bar{a}, \bar{b} \in M$ with $\bar{a} \neq \bar{b}$,

$$\lambda(\mathcal{O}(\bar{a}) - \mathcal{O}(\bar{b})) > \lambda(\bar{a} - \bar{b}).$$

**Remark:** Practically this means that after application of $\mathcal{O}$, the derivatives in $\bar{a}$ and $\bar{b}$ agree to a higher order than before application of $\mathcal{O}$.

**Example:** The antiderivation $\partial_k^{-1}$.

**Theorem (DA Fixed Point Theorem)** Let $\mathcal{O}$ be a contracting operator on $M \subset \mathcal{D}_v^D$ that maps $M$ into $M$. Then $\mathcal{O}$ has a unique fixed point $\bar{a} \in M$ that satisfies the fixed point problem

$$\bar{a} = \mathcal{O}(\bar{a}).$$

Moreover, let $a_0$ be any element in $M$. Then the sequence

$$a_k = \mathcal{O}(a_{k-1}) \text{ for } k = 1, 2, \ldots$$

converges in finitely many steps, at most $(n + 1)$ steps, to the fixed point $\bar{a}$. 
DA Fixed Point PDE Solvers

The DA fixed point theorem allows to solve PDEs iteratively in finitely many steps by rephrasing them in terms of a fixed point problem.

Consider the rather general PDE

\[ a_1 \frac{\partial}{\partial x} \left( a_2 \frac{\partial}{\partial x} V \right) + b_1 \frac{\partial}{\partial y} \left( b_2 \frac{\partial}{\partial y} V \right) + c_1 \frac{\partial}{\partial z} \left( c_2 \frac{\partial}{\partial z} V \right) = 0, \]

where \( a_i, b_i, c_i \) are functions of \( x, y, z \).

The PDE is re-written as

\[
V = V|_{y=0} + \int_0^y \left. \frac{1}{b_2} \left( b_2 \frac{\partial V}{\partial y} \right) \right|_{y=0} dy - \int_0^y \int_0^y \left( \frac{a_1}{b_1} \frac{\partial}{\partial x} \left( a_2 \frac{\partial V}{\partial x} \right) + \frac{c_1}{b_1} \frac{\partial}{\partial z} \left( c_2 \frac{\partial V}{\partial z} \right) \right) dy dy,
\]

in fixed point form.

Assume the derivatives of \( V \) and \( \partial V/\partial y \) with respect to \( x \) and \( z \) are known in the plane \( y = 0 \). Then the right hand side is contracting with respect to \( y \), and the various orders in \( y \) can be iteratively calculated by mere iteration.
Analytical Field on Axis for a thick Solenoid

\[ B_s(s) = \frac{\mu_0 I_n}{2(R_2-R_1)} \left\{ s \log \left( \frac{R_2+\sqrt{R_2^2+s^2}}{R_1+\sqrt{R_1^2+s^2}} \right) - (s-l) \log \left( \frac{R_2+\sqrt{R_2^2+(s-l)^2}}{R_1+\sqrt{R_1^2+(s-l)^2}} \right) \right\} \]

\[ V(s) = \frac{\mu_0 I_n}{4(R_2-R_1)} \left\{ s^2 \log \left( \frac{R_2+\sqrt{R_2^2+s^2}}{R_1+\sqrt{R_1^2+s^2}} \right) - (s-l)^2 \log \left( \frac{R_2+\sqrt{R_2^2+(s-l)^2}}{R_1+\sqrt{R_1^2+(s-l)^2}} \right) \right\} + R_2\sqrt{R_2^2+s^2} - R_1\sqrt{R_1^2+s^2} - R_2\sqrt{R_2^2+(s-l)^2} + R_1\sqrt{R_1^2+(s-l)^2} \]
Balbekov Ring: Short Section

Axial Field
1) Transfer Map with **Kick** Approximation in the **Asymptotic** Fields

<table>
<thead>
<tr>
<th>Transfer Value</th>
<th>1.015304</th>
<th>0.3060451E-11</th>
<th>0.2022730E-03</th>
<th>0.5845774E-04</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.968705</td>
<td>1.388108</td>
<td>1.284507</td>
<td>1.099203</td>
<td>5000</td>
<td></td>
</tr>
<tr>
<td>-12.74829</td>
<td>-1.191336</td>
<td>1.970436</td>
<td>4.189763</td>
<td>4100</td>
<td></td>
</tr>
<tr>
<td>-22.23560</td>
<td>-5.817135</td>
<td>-0.2996768</td>
<td>5.761626</td>
<td>3200</td>
<td></td>
</tr>
</tbody>
</table>

* The consistency was checked with linear matrices supplied by Balbekov.

2) Transfer Map with **Kick** Approximation in the **Realistic** Fields

<table>
<thead>
<tr>
<th>Transfer Value</th>
<th>0.3762572E-01</th>
<th>0.1324007E-03</th>
<th>0.9123892E-05</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.092008</td>
<td>0.3762572E-01</td>
<td>0.1216993E-10</td>
<td>0.1324008E-03</td>
<td>0100</td>
</tr>
<tr>
<td>-0.1324007E-03</td>
<td>-0.9123892E-05</td>
<td>0.3762572E-01</td>
<td>-0.9144481</td>
<td>0010</td>
</tr>
<tr>
<td>-0.1216993E-10</td>
<td>-0.1324008E-03</td>
<td>1.092008</td>
<td>0.3762572E-01</td>
<td>0001</td>
</tr>
</tbody>
</table>

* Omitted ...

<table>
<thead>
<tr>
<th>Transfer Value</th>
<th>2.194351</th>
<th>1.925709</th>
<th>2.734348</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.100432</td>
<td>1.247951</td>
<td>1.790312</td>
<td>4100</td>
<td></td>
</tr>
<tr>
<td>9.421870</td>
<td>-4.741542</td>
<td>-0.5027561</td>
<td>3200</td>
<td></td>
</tr>
</tbody>
</table>

3) Transfer Map Computed with **Fringe** Fields by **COSY**

<table>
<thead>
<tr>
<th>Transfer Value</th>
<th>0.9113584E-02</th>
<th>0.9101484</th>
<th>0.2707143E-04</th>
<th>0.8741688E-06</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.098631</td>
<td>0.9113584E-02</td>
<td>0.1597370E-05</td>
<td>0.2707097E-04</td>
<td>0100</td>
<td></td>
</tr>
<tr>
<td>-0.2707143E-04</td>
<td>0.8741688E-06</td>
<td>0.9113584E-02</td>
<td>-0.9101484</td>
<td>0010</td>
<td></td>
</tr>
<tr>
<td>0.1597370E-05</td>
<td>-0.2707097E-04</td>
<td>1.098631</td>
<td>0.9113584E-02</td>
<td>0001</td>
<td></td>
</tr>
</tbody>
</table>

* Omitted ...

<table>
<thead>
<tr>
<th>Transfer Value</th>
<th>1.095873</th>
<th>0.6296333</th>
<th>1.603693</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.302134</td>
<td>-0.2037065</td>
<td>0.6387705</td>
<td>4100</td>
<td></td>
</tr>
<tr>
<td>3.442199</td>
<td>-3.197581</td>
<td>-1.227331</td>
<td>3200</td>
<td></td>
</tr>
</tbody>
</table>
Muon Beam Ring Cooler by V. Balbekov
(Pictures: Courtesy of Balbekov)

Long Section

198 cm  272 cm  198 cm
10.5 cm  R 81 cm
3.5 cm

668 cm

J = 43.79 A/mm²

Axial Field with Infinitely Extended End Coils
Balbekov Ring: Long Section

Axial Field

Bz (T)

s (m)
Balbekov's Kick Approximation V.S. COSY Computation
Long Section (Length 6.68m, Inner radius 81cm, Coil thickness 3.5cm to 10.5cm)

Linear Transfer Map with Balbekov's Kick Approximation (Full Aperture)

-0.1563521E-02  0.4816040E-01 -0.3071766E-01  0.9461698        1000
-0.5360646E-01 -0.1563561E-02 -1.053164        -0.3071766E-01        0100
0.3071766E-01 -0.9461698        -0.1563521E-02  0.4816040E-01        0010
1.053164        0.3071766E-01 -0.5360646E-01 -0.1563561E-02        0001

Linear Transfer Map Computed with Fringe Fields by COSY (Full Aperture)

0.2349601        0.7548866        0.8860107E-01  0.2030465        1000
-1.157433        0.2477034        -0.3113219        0.4122390E-01        0100
-0.8860107E-01 -0.2030465        0.2349601        0.7548866        0010
0.3113219        -0.4122390E-01 -1.157433        0.2477034        0001

Difference between the Kick Approximation and COSY with Smaller Aperture
Traces of 1/2 Ring with Various Solenoid Models (Bending: Hard Edge Model, B=1, angle=45°)

Correct Fringe Fields

Hard Edge, Finely Long

Hard Edge, Infinitely Long
7-th order, 100 1/2 turns, CET=250 IC FR 2 Expo

Defining:

Ground: Hard Edge (Infini-Tip Long)

O.100

O.200
Bending: Hard Edge

Second: Hard Edge (Infinity Long)

7th order, 100 1/2 turns, Bext=49 IX PR 0 EXPO
Bending: with Fringe Fields
Solenoid: Hand Edge (Infinity Long)
7-th order, 100 1/2 turns, Extc=249, Ic FR 2 EXPO
Conclusion and Outlook

- There are strong resonance structures with/without FF.
- The FF effects have a dramatic impact on performance.

Next studies....

- What is the impact of damping on the performance.