Calculated life of π/μ beams in a decay channel

(Longitudinal Motion)

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Integral expressions giving densities of pion and muon bunches in a decay channel are derived from the kinematics, and compared with Monte Carlo simulations of beam transport.

Goal :

- compete with MC methods (easy : so slow !)
- provide means of understanding, of estimates and for checking validity of MC methods
- review various processes of concern in the $\pi \to \mu + \nu$ decay,

$\to$ angle, momentum, time distributions, how they build up in the course of the decay

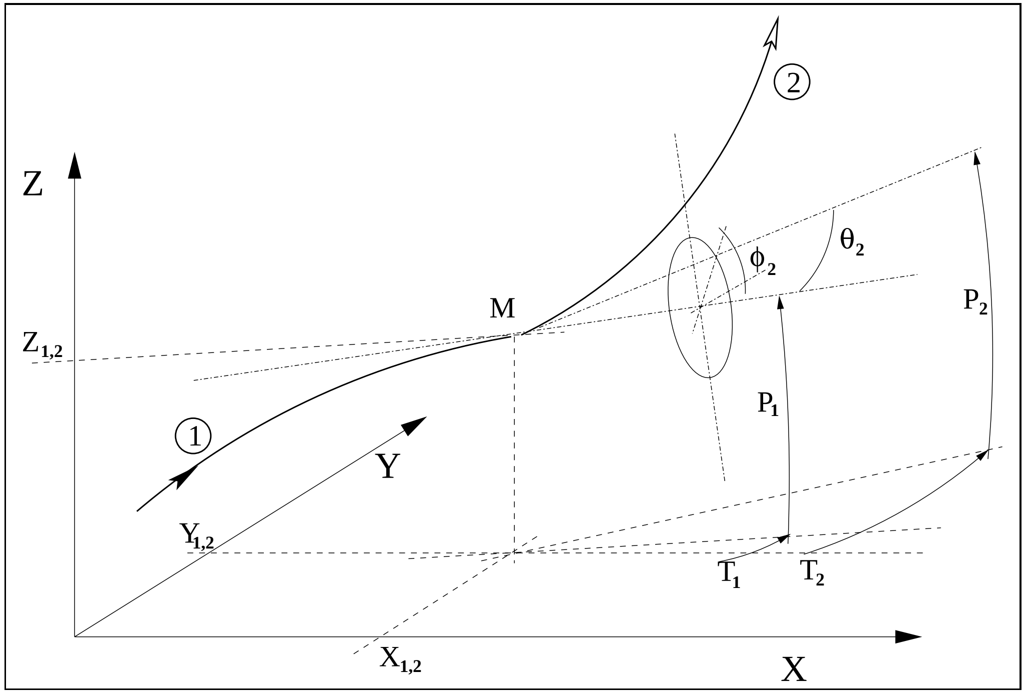
- understand typical values of relevant parameters

Contents

1 Working hypothesis (Kinematical data)

Density calculations to follow are based on the kinematical relations of the Z-decay process

$$\pi \to \mu + \nu$$



$(1) \to (2)$ decay.

• Basic ingredients are :
pion mass m_π, com lifetime τ_π^*, energy E_π, muon mass m_μ, decay angle θ_μ^*,

• from what one gets... a few more, basic ingredients we need in what follows

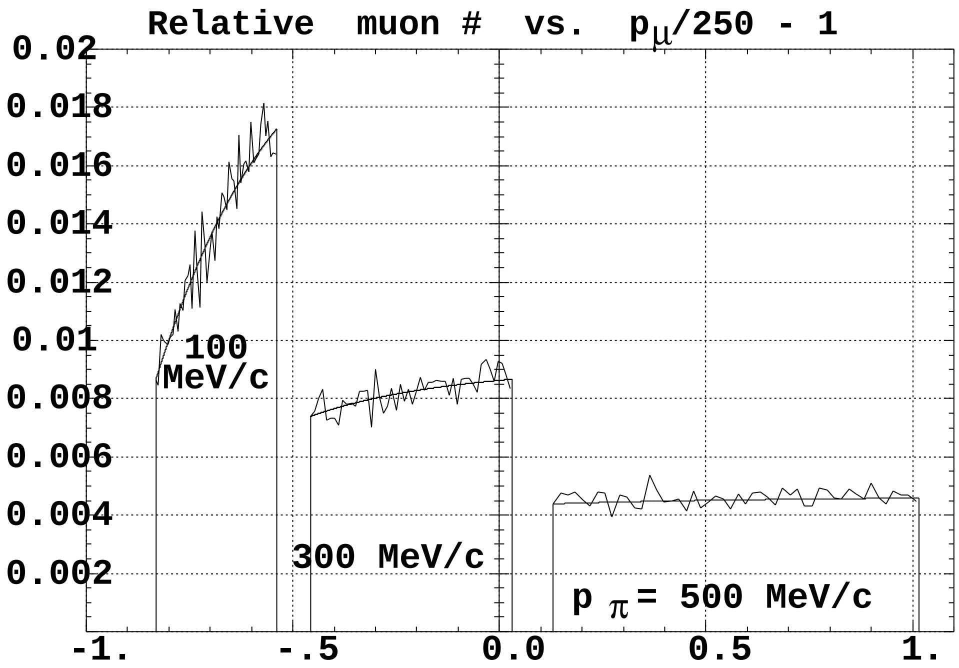
- energy of muon in com $\quad E_\mu^* = (m_\pi^2 + m_\mu^2)/2m_\pi$
- energy of muon in lab $\quad E_\mu = \gamma_\pi \left(E_\mu^* + \beta_\pi p_\mu^* \cos\theta_\mu^* \right)$
- lab lifetime $\quad \tau_\pi = \gamma_\pi \tau_\pi^*$
- Parameters of exponential decay law $\quad N/N_0 = e^{-s/s_\pi}, \quad s_\pi = \beta_\pi \gamma_\pi c \tau_\pi^*$
- lab decay angles $\quad \phi_\mu = \phi_\mu^*, \qquad \tan\theta_\mu = \frac{1}{\gamma_\pi} \frac{\sin\theta_\mu^*}{\beta_\pi/\beta_\mu^* + \cos\theta_\mu^*}$

2 Muon momentum density

The muon momentum density $g_{p_\mu|p_\pi}$ is p_π-conditional and satisfies

$$\boxed{g_{p_\mu|p_\pi}(p_\mu) = f\left[\theta^*_\mu(p_\mu)\right]\ \left|\frac{d\theta^*_\mu}{dp_\mu}\right| = \frac{m_\pi}{2p_\pi p^*_\mu}\, p^2_\mu/\sqrt{p^2_\mu + m^2_\mu} \quad (\theta^*_\mu \in [0,\pi])},$$ wherein

- $f(\theta^*_\mu) = \sin\theta^*_\mu/2 \quad (\theta^*_\mu \in [0,\pi])$ is the density of θ^*_μ
- $\theta^*_\mu(p_\mu)$ is the root of $E_\mu = \gamma_\pi\left(E^*_\mu + \beta_\pi p^*_\mu \cos\theta^*_\mu\right)$ for given value p_μ.



$g_{p_\mu|p_\pi}(p_\mu)$, for $p_\pi = 10,\ 300$ and 500 MeV/c.
Side limits satisfy $\hat{p}_\mu = \gamma_\pi(\beta_\pi E^*_\mu \mp p^*_\mu)$.
Superimposed Monte Carlo histograms are for 10^4 initial pions

3 Parent pion bunch spectrum

The probability density of decay as a function of flight distance s, given the pion momentum p_π writes :

$$g_{s|p_\pi}(s, p_\pi) = \frac{\lambda}{p_\pi} \exp(-\frac{\lambda s}{p_\pi}) \qquad (\lambda = m_\pi / c\tau_\pi^*)$$

Given a parent pion bunch with known initial momentum density $g_{p_\pi}(p_\pi)$ one gets the 2-D density

$$g_{s,p_\pi}(s, p_\pi) = g_{s|p_\pi} \times g_{p_\pi} \qquad (\text{such that } \int_{s=0}^{\infty} \int_{p_\pi}^{p_{1\mu}} g_{s,p_\pi}(s, p_\pi)\, ds\, dp_\pi = 1)$$

- We simplify by taking uniform initial pion momentum density

$$g_{p_\pi}(p_\pi) = \mathbf{1}_{\Delta p_\pi}(p_\pi) = \frac{1}{p_{\pi_2} - p_{\pi_1}} \qquad \textit{iff}\ p \in [p_{\pi_1}, p_{\pi_2}]$$

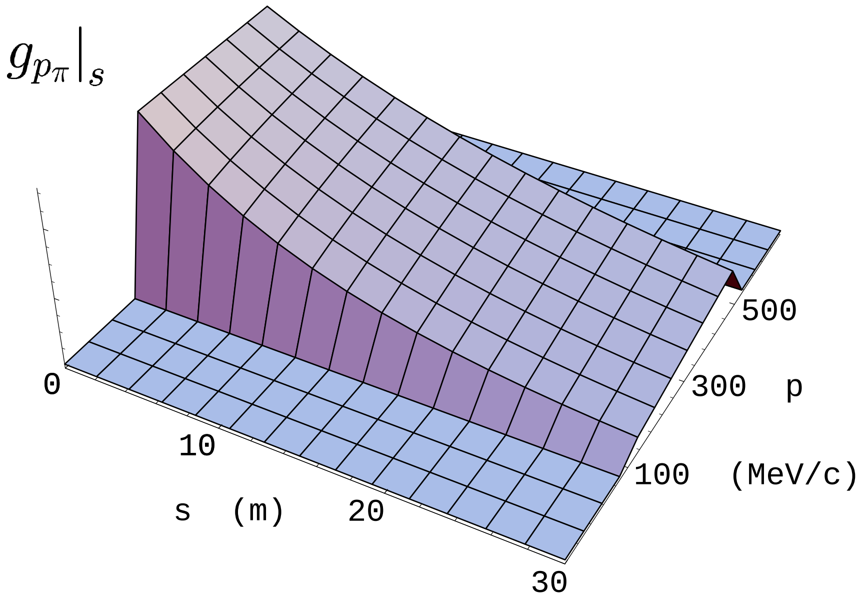
- and integrate *wrt.* s, from 0 to s
- which yields the p_π-density of the decayed parent pions at distance s

$$g_{p_\pi}(p_\pi)\big|_s = \mathbf{1}_{\Delta p_\pi}(p_\pi)\,(1 - \exp(-\frac{\lambda s}{p_\pi}))$$

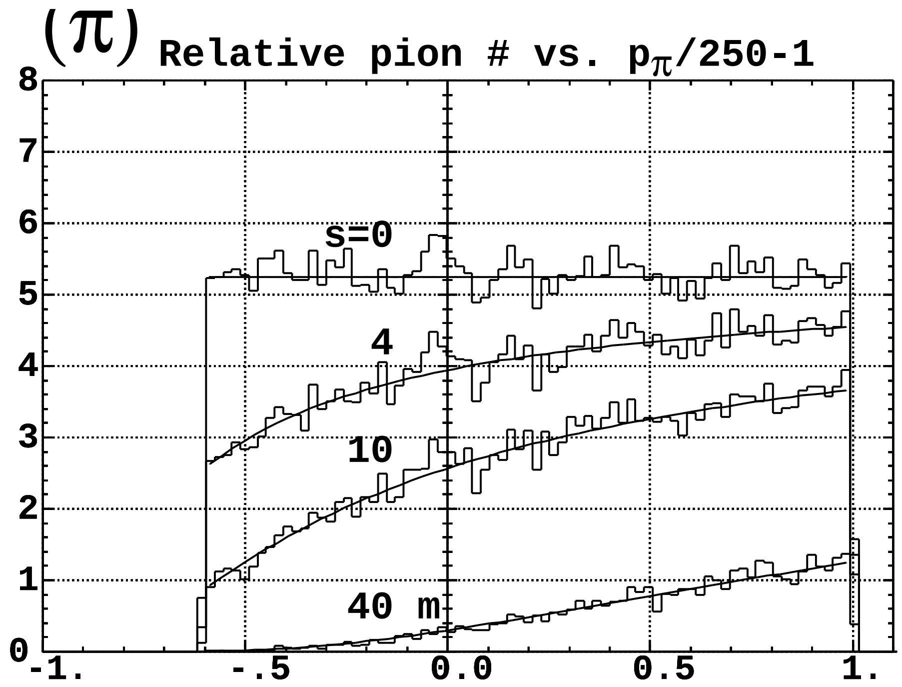
The p_π-density of the *non-decayed* pion population at given s ensues,

$$\boxed{\bar{g}_{p_\pi}(p_\pi)\Big|_s = (\mathbf{1}_{\Delta p_\pi}(p_\pi) - g_{p_\pi}(p_\pi)\Big|_s) = \mathbf{1}_{\Delta p_\pi}(p_\pi)\ \exp(-\frac{\lambda s}{p_\pi})}$$

Figure below : case of a pion bunch launched at $s = 0$ with zero size and with momentum $p_\pi \in [100, 500]$ MeV/c uniform



Pion momentum density as a function of distance along the decay channel



s-sections.

4 Muon bunch spectrum

- The muon bunch momentum distribution at s has the density $g_{p_\mu}(p_\mu)\big|_s = g_{p_\mu|p_\pi}(p_\mu) \times g_{p_\pi}(p_\pi)\big|_s$
  - Integration over the initial momentum bite $[p_{\pi_1}, p_{\pi_2}]$ of the pion bunch yields the left graphs below.

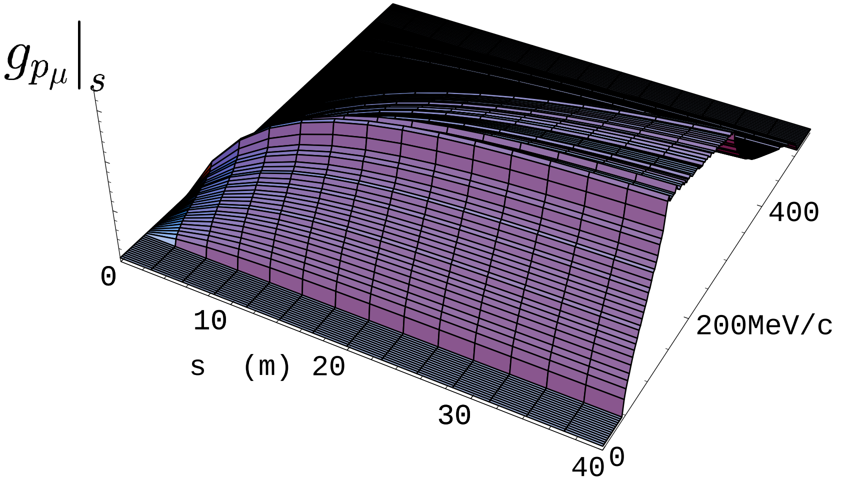
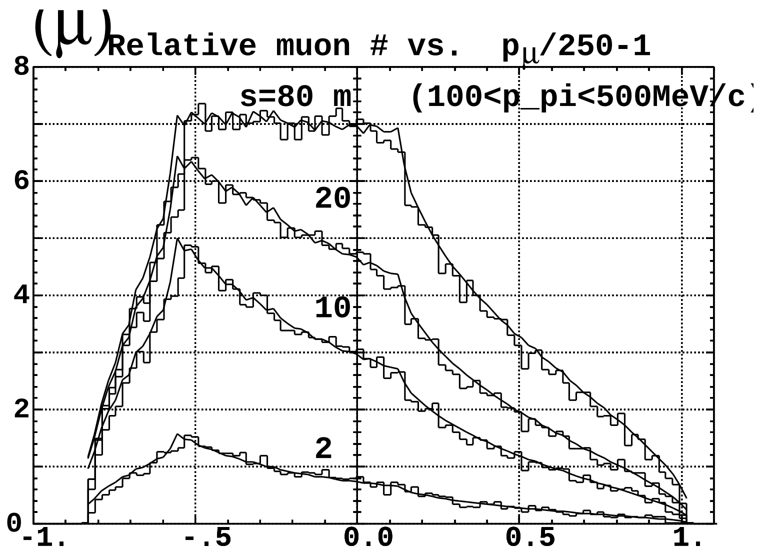
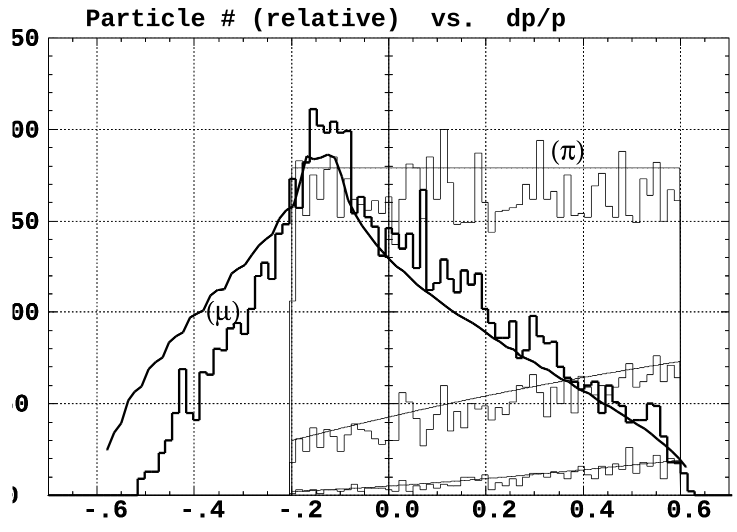


Fig. below, <u>case of a real AG line</u> [Autin, these Procs.], including $\Phi 60$ cm collimation, $p_\pi \in [200, 400]$ MeV/c : low E muons are lost.





5 Decay rate

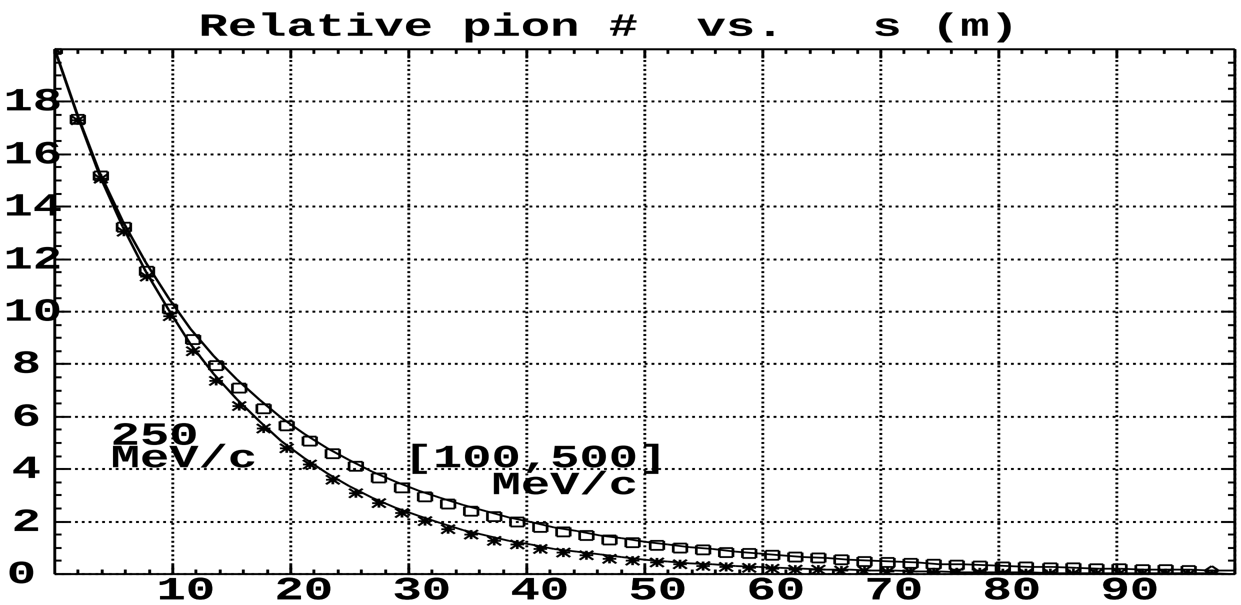
Integration *wrt.* p_π yields the number of survived pions N_π and of created muons N_μ vs. distance s,

- $N_\pi(s) = N_0 \int_{p_{\pi_1}}^{p_{\pi 2}} \exp(-\lambda s/p)\, dp \;/\; (p_{\pi_2} - p_{\pi_1})$

$$= N_0 \left(\frac{p_{\pi_2} + \lambda\, s\, e^{\lambda\, s/p_{\pi 2}}\, \mathrm{Ei}(-\lambda\, s/p_{\pi_2})}{e^{\lambda\, s/p_{\pi 2}}} - \frac{p_{\pi_1} + \lambda\, s\, e^{\lambda\, s/p_{\pi 1}}\, \mathrm{Ei}(-\lambda\, s/p_{\pi_1})}{e^{\lambda\, s/p_{\pi 1}}} \right) /(p_{\pi_2} - p_{\pi_1})$$

- $N_\mu(s) = 1 \;-\; N_\pi(s)$

Figure below :
case of 250 MeV/c mono-momentum,
and case of uniform $p_\pi \in [100, 500]$ MeV/c parent pion bunch
$\Rightarrow$ a large momentum bite has some effect on $N_{\pi,\mu}(s)$.



Markers are from MC.

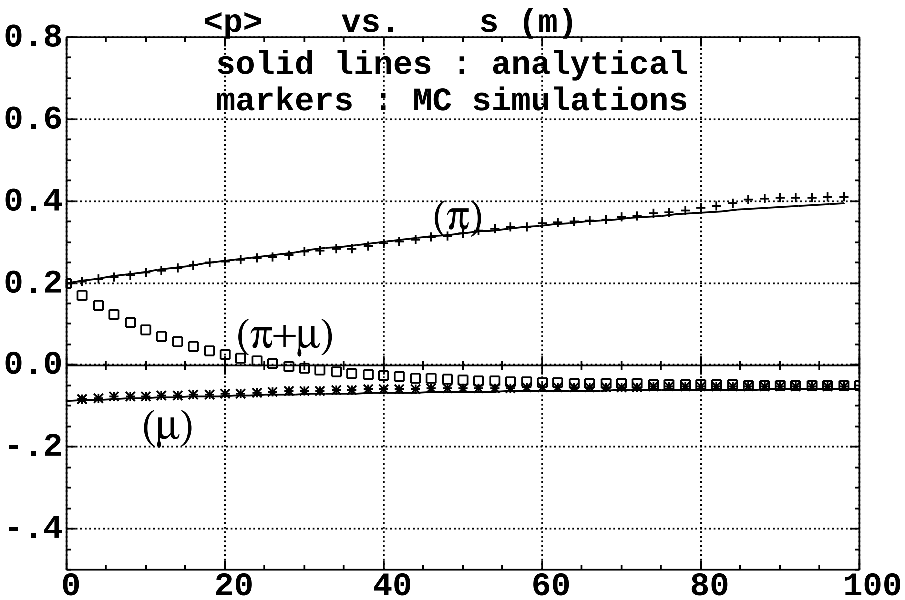
6 Average momentum of π and μ beams

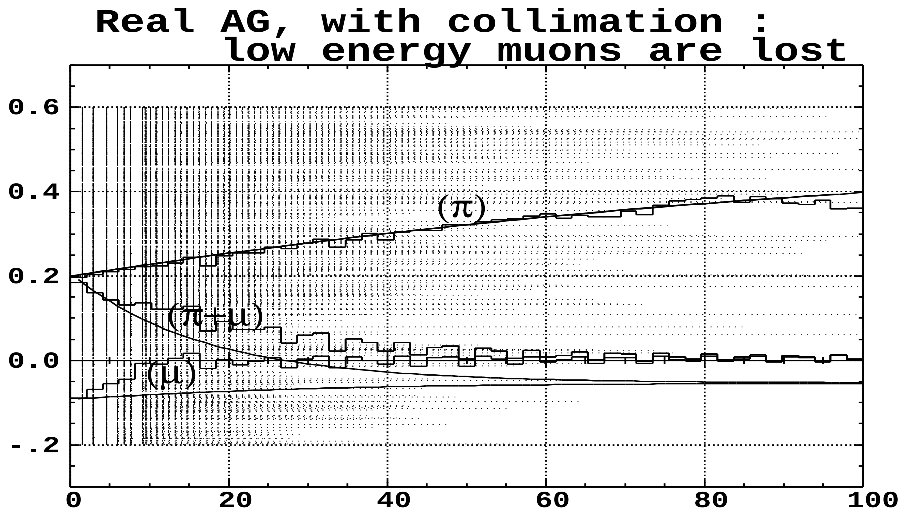
Parent pion beam : $\bar{p}_\pi(s) = \int_{p_{\pi_1}}^{p_{\pi_2}} p\, \exp(-\lambda s/p)\, dp \;/\; \int_{p_{\pi_1}}^{p_{\pi_2}} \exp(-\lambda s/p)\, dp$

$$= \sum_{i=1,2} (-)^i \frac{p_{\pi_i}^2 - \lambda\, s\, p_{\pi_i} - \lambda^2\, s^2\, e^{\lambda\, s/p_{\pi_i}}\, \mathrm{Ei}(-\lambda\, s/p_{\pi_i})}{2\, e^{\lambda\, s/p_{\pi_i}}} \Big/ \sum_{i=1,2} (-)^i \frac{p_{\pi_i} + \lambda\, s\, e^{\lambda\, s/p_{\pi_i}}\, \mathrm{Ei}(-\lambda\, s/p_{\pi_i})}{e^{\lambda\, s/p_{\pi_i}}}$$

Muon beam : $\bar{p}_\mu(s) = \int_{p_{\mu_1}}^{p_{\mu_2}} p\, \delta N_{p_\mu}(s,p) dp \;/\; \int_{p_{\mu_1}}^{p_{\mu_2}} \delta N_{p_\mu}(s,p) dp$

This is represented in Fig. below (solid curves ; markers are from MC)





Average momenta of both pion and muon beams are increasing functions of the distance, because the lower energy parent pions decay faster. The average momentum of the $\pi + \mu$ beam decreases instead, from its upper value that coincides with the parent beam one's at start, to its lower value that coincides with the muon beam one's when all parents have decayed.

$\Rightarrow$ Suggests that gradients along the π/μ collect channel should track momentum (as in RLA's) : true !, best transmission is obtained that way in the AG channel [this workshop].

7 Longitudinal phase space

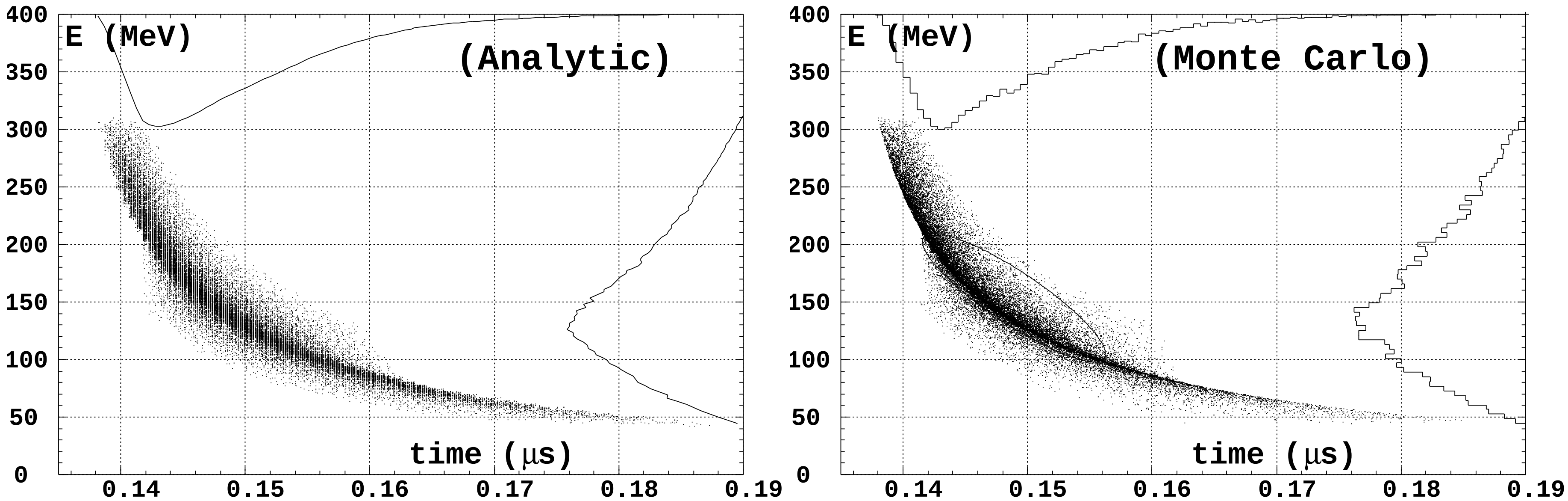
• The time distribution of a muon bunch, as observed at distance s, assuming all muons are created at given $s_d < s$ is obtained by a change of variable $p_\mu \rightarrow t_\mu : t_\mu = (s - s_d)/\beta_\mu c$, yielding

$$g_{t_\mu}(t_\mu)\big|_{s_d} = g_{t_\mu|p_\pi}(t_\mu) \ \times \ g_{p_\pi}(p_\pi)\big|_{s=s_d} \tag{1}$$

• Taking into account the pion density at s one gets

$$g_{t_\mu}(t_\mu) = g_{t_\mu|p_\pi}(t_\mu) \times g_{p_\pi}(p_\pi)\big|_{s_d} \times g_{s|p_\pi}(s, p_\pi) \tag{2}$$

• A particle population can be reconstructed, left Fig. below.



Muon longitudinal phase space at $s = 40$ m.

Mathematica functions have been built : TimeDistribution[s,$E_{\pi,min}$, $E_{\pi,max}$, *Options*], EnergyDistribution[s,$E_{\pi,min}$, $E_{\pi,max}$, *Options*], LongitudinalDistribution[s,$E_{\pi,min}$, $E_{\pi,max}$, *Options*].

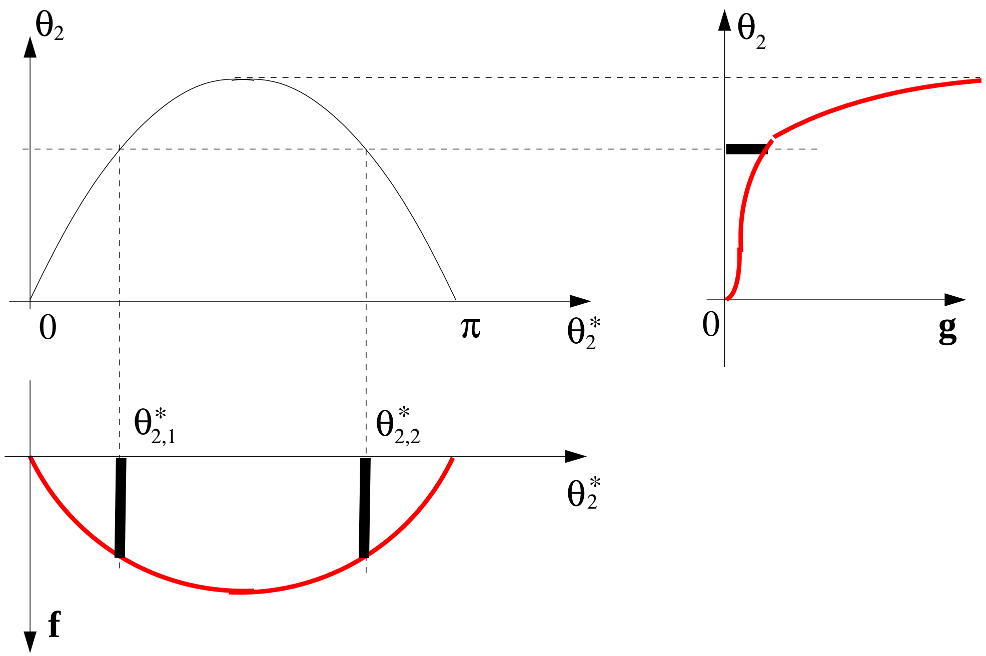
8 Next step : transverse phase space

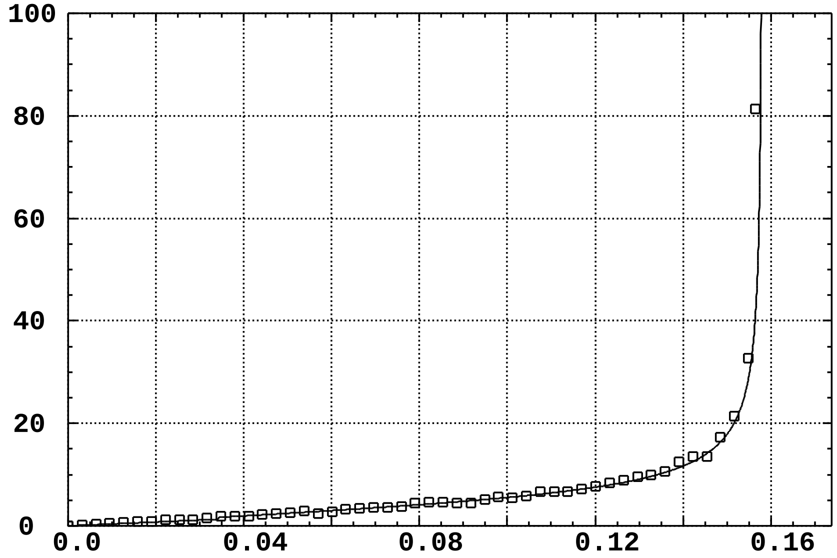
8.1 Step 1 : decay angle

The density of lab decay angle θ_μ is p_π-conditional and satisfies

$$\boxed{g_{\theta_\mu|p_\pi}(\theta_\mu) = \sum_{i=1,2} f(\theta^*_{\mu,i})|d\theta^*_\mu/d\theta_\mu|_i = \frac{\gamma_\pi^2}{2}\tan\theta_\mu \left|\sum_{i=1,2}(-)^{i+1}\frac{(\beta_\pi/\beta^*_\mu+\cos\theta^*_{\mu,i})^3}{1+\beta_\pi/\beta^*_\mu\,\cos\theta^*_{\mu,i}}\right|}$$

- with $\theta^*_{\mu,i}$ (i=1,2) the two roots of $\tan\theta_\mu = \frac{1}{\gamma_\pi}\frac{\sin\theta^*_\mu}{\beta_\pi/\beta^*_\mu+\cos\theta^*_\mu}$





$g_{\theta_\mu|p_\pi}(\theta_\mu)$ for $p_\pi = 250$ MeV/c.

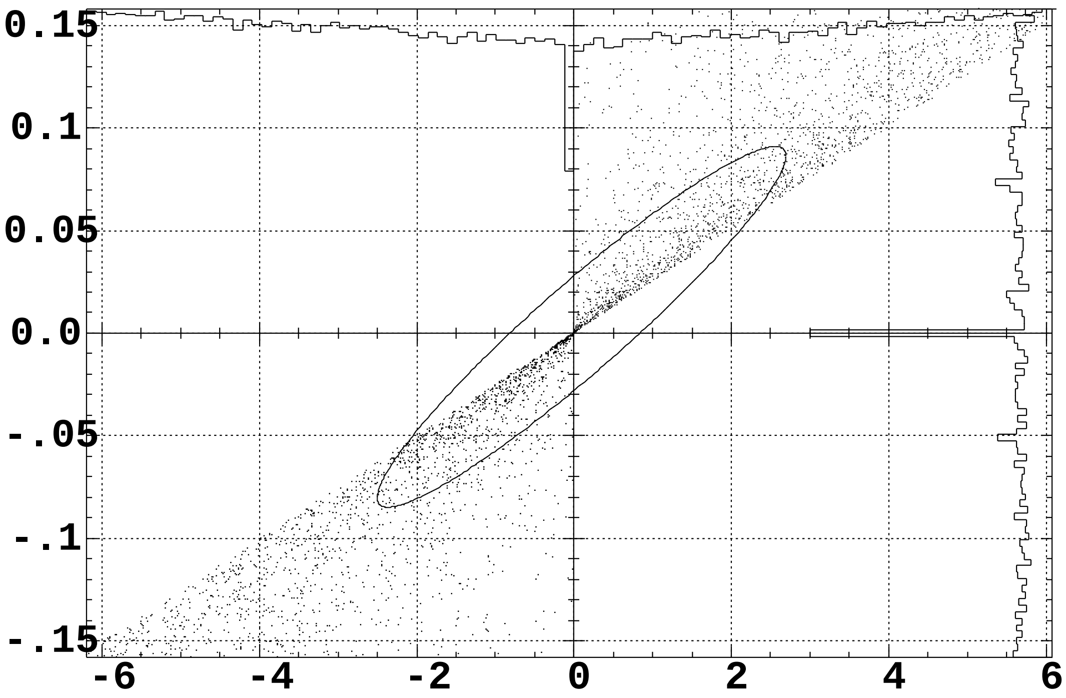
Limit decay angles values satisfy $\hat{\theta}_\mu = \pm\mathrm{arctg}\left(p^*_\mu m_\pi/\sqrt{(p_\pi^2 m_\mu^2 - p^{*2}_\mu m_\pi^2}\right)$ as expected.

8.2 Step 2 : get the "target effect" (transverse phase space)

The projection of θ_μ onto x and x' or onto z and z' can be derived from

$$\frac{\vec{\beta}_\mu}{|\beta_\mu|} = \begin{bmatrix} \cos x'_\pi & -\sin x'_\pi & 0 \\ \sin x'_\pi & \cos x'_\pi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos z'_\pi & 0 & -\sin z'_\pi \\ 0 & 1 & 0 \\ \sin z'_\pi & 0 & \cos z'_\pi \end{bmatrix} \begin{bmatrix} \cos\theta_\mu \\ \sin\theta_\mu cos\phi_\mu \\ \sin\theta_\mu \sin\phi_\mu \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{bmatrix} \quad (3)$$

We therefore know the change of variables $(\theta_\mu, \phi_\mu) \to (x, x')$, and $(\theta_\mu, \phi_\mu) \to (z, z')$.
From what we can expect to draw the the densities below, as obtained from Monte Carlo simulations :



$x - x'$ muon phase space, s=40 meters downstream of production target. The parent pion bunch is on-axis, zero transverse size, $p_\pi = 250$ MeV/c.

Eps/pi, Beta, Alpha: 7.2685E-02 92.9 -2.98 (MKSA)

in progress...

9 Conclusions

1/ Nothing precludes replacing the uniform pion momentum distribution $\mathbf{1}_{\Delta p_\pi}(p_\pi)$ used here for illustration, by more realistic ones, which will be undertaken in the frame of the decay channel studies.

2/ Next to come : second order transport of these densities through AG structures.