Calculated life of $\pi/\mu$ beams in a decay channel

(Longitudinal Motion)

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Integral expressions giving densities of pion and muon bunches in a decay channel are derived from the kinematics, and compared with Monte Carlo simulations of beam transport.

Goal:
- compete with MC methods (easy : so slow !)
- provide means of understanding, of estimates and for checking validity of MC methods
- review various processes of concern in the $\pi \rightarrow \mu + \nu$ decay,
- angle, momentum, time distributions, how they build up in the course of the decay
- understand typical values of relevant parameters
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1 Working hypothesis (Kinematical data)

Density calculations to follow are based on the kinematical relations of the Z-decay process

\[ \pi \rightarrow \mu + \nu \]

- Basic ingredients are:
  - pion mass \( m_\pi \), com lifetime \( \tau^*_\pi \), energy \( E_\pi \), muon mass \( m_\mu \), decay angle \( \theta^*_\mu \),
  - from what one gets... a few more, basic ingredients we need in what follows
    - energy of muon in \( \text{com} \)
      \[ E^*_\mu = (m_\pi^2 + m_\mu^2)/2m_\pi \]
    - energy of muon in \( \text{lab} \)
      \[ E_\mu = \gamma_\pi (E^*_\mu + \beta_\pi p^*_\mu \cos \theta^*_\mu) \]
    - \( \text{lab} \) lifetime
      \[ \tau_\pi = \gamma_\pi \tau^*_\pi \]
    - Parameters of exponential decay law
      \[ N/N_0 = e^{-s/s_\pi}, \quad s_\pi = \beta_\pi \gamma_\pi c \tau^*_\pi \]
    - \( \text{lab} \) decay angles
      \[ \phi_\mu = \phi^*_\mu, \quad \tan \theta_\mu = \frac{1}{\gamma_\pi \beta_\pi / \beta^*_\mu + \cos \theta^*_\mu} \sin \theta^*_\mu \]
2 Muon momentum density

The muon momentum density $g_{p_{\mu}|p_\pi}$ is $p_\pi$-conditional and satisfies

$$g_{p_{\mu}|p_\pi}(p_\mu) = f \left[ \theta^*_\mu(p_\mu) \right] = \frac{d\theta^*_\mu}{dp_\mu} = \frac{m_\pi}{2p_\pi p_\mu^*} \frac{p_\mu^2}{\sqrt{p_\mu^2 + m_\mu^2}} \left( \theta^*_\mu \in [0, \pi] \right),$$

wherein

- $f(\theta^*_\mu) = \sin \theta^*_\mu / 2$ $(\theta^*_\mu \in [0, \pi])$ is the density of $\theta^*_\mu$
- $\theta^*_\mu(p_\mu)$ is the root of $E_\mu = \gamma_\pi \left( E^*_\mu + \beta_\pi p_\mu^* \cos \theta^*_\mu \right)$ for given value $p_\mu$.

$g_{p_{\mu}|p_\pi}(p_\mu)$, for $p_\pi = 10$, 300, and 500 MeV/c.

Side limits satisfy $\hat{p}_\mu = \gamma_\pi (\beta_\pi E^*_\mu \mp p_\mu^*)$.

Superimposed Monte Carlo histograms are for $10^4$ initial pions.
3 Parent pion bunch spectrum

The probability density of decay as a function of flight distance $s$, given the pion momentum $p_\pi$ writes:

$$g_{s|p\pi}(s, p_\pi) = \frac{\lambda}{p_\pi} \exp\left(-\frac{\lambda s}{p_\pi}\right) \quad (\lambda = m_\pi/c\tau_{\pi}^*)$$

Given a parent pion bunch with known initial momentum density $g_{p\pi}(p_\pi)$ one gets the 2-D density

$$g_{s,p\pi}(s, p_\pi) = g_{s|p\pi} \times g_{p\pi} \quad \text{(such that } \int_{s=0}^{\infty} \int_{p_\pi}^{p_{1\mu}} g_{s,p\pi}(s, p_\pi) \, ds \, dp_\pi = 1)$$

- We simplify by taking uniform initial pion momentum density

$$g_{p\pi}(p_\pi) = 1_{\Delta p\pi}(p_\pi) = \frac{1}{p_{\pi2} - p_{\pi1}} \quad \text{iff } p \in [p_{\pi1}, p_{\pi2}]$$

- and integrate wrt. $s$, from 0 to $s$

- which yields the $p_\pi$-density of the decayed parent pions at distance $s$

$$g_{p\pi}(p_\pi)|_s = 1_{\Delta p\pi}(p_\pi) \left(1 - \exp\left(-\frac{\lambda s}{p_\pi}\right)\right)$$
The $p_\pi$-density of the non-decayed pion population at given $s$ ensues,

$$\bar{g}_{p_\pi}(p_\pi)\bigg|_s = \left(1\Delta_{p_\pi}(p_\pi) - g_{p_\pi}(p_\pi)\bigg|_s\right) = 1\Delta_{p_\pi}(p_\pi) \exp\left(-\frac{\lambda s}{p_\pi}\right)$$

Figure below: case of a pion bunch launched at $s = 0$ with zero size and with momentum $p_\pi \in [100, 500]$ MeV/c uniform

Pion momentum density as a function of distance along the decay channel
4 Muon bunch spectrum

- The muon bunch momentum distribution at $s$ has the density $g_{p\mu}(p_\mu)|_s = g_{p\mu}(p_\mu) \times g_{p\pi}(p_\pi)|_s$
- Integration over the initial momentum bite $[p_{\pi_1}, p_{\pi_2}]$ of the pion bunch yields the left graphs below.

Fig. below, case of a real AG line [Autin, these Procs.], including $\Phi 60$ cm collimation, $p_\pi \in [200, 400]$ MeV/c:
low E muons are lost.
5 Decay rate

Integration wrt. $p_\pi$ yields the number of survived pions $N_\pi$ and of created muons $N_\mu$ vs. distance $s$,

- $N_\pi(s) = N_0 \int_{p_{\pi_1}}^{p_{\pi_2}} \exp(-\lambda s/p) \, dp / (p_{\pi_2} - p_{\pi_1})$
  
  $$= N_0 \left( \frac{p_{\pi_2} \exp(-\lambda s/p_{\pi_2})}{e^{\lambda s/p_{\pi_2}}} - \frac{p_{\pi_1} \exp(-\lambda s/p_{\pi_1})}{e^{\lambda s/p_{\pi_1}}} \right) / (p_{\pi_2} - p_{\pi_1})$$

- $N_\mu(s) = 1 - N_\pi(s)$

Figure below:

case of 250 MeV/c mono-momentum, and case of uniform $p_\pi \in [100, 500]$ MeV/c parent pion bunch

$\Rightarrow$ a large momentum bite has some effect on $N_{\pi,\mu}(s)$. 
Average momentum of $\pi$ and $\mu$ beams

**Parent pion beam**:

$$\bar{p}_\pi(s) = \frac{\int_{p_{\pi 1}}^{p_{\pi 2}} p \exp(-\lambda s / p) \, dp}{\int_{p_{\pi 1}}^{p_{\pi 2}} \exp(-\lambda s / p) \, dp}$$

$$= \sum_{i=1,2} \left( -\right)^i \frac{p_{\pi i}^2 - \lambda s p_{\pi i} - \lambda^2 s^2 \exp(s / p_{\pi i})}{2 \exp(s / p_{\pi i})}$$

**Muon beam**:

$$\bar{p}_\mu(s) = \frac{\int_{p_{\mu 1}}^{p_{\mu 2}} p \delta N_{p_{\mu}}(s, p) \, dp}{\int_{p_{\mu 1}}^{p_{\mu 2}} \delta N_{p_{\mu}}(s, p) \, dp}$$

This is represented in Fig. below (solid curves; markers are from MC)

![Graph](image)

Average momenta of both pion and muon beams are increasing functions of the distance, because the lower energy parent pions decay faster. The average momentum of the $\pi + \mu$ beam decreases instead, from its upper value that coincides with the parent beam one’s at start, to its lower value that coincides with the muon beam one’s when all parents have decayed.

⇒ Suggests that gradients along the $\pi/\mu$ collect channel should track momentum (as in RLA’s): true!, best transmission is obtained that way in the AG channel [this workshop].
7 Longitudinal phase space

- The time distribution of a muon bunch, as observed at distance $s$, assuming all muons are created at given $s_d < s$ is obtained by a change of variable $p_\mu \rightarrow t_\mu : t_\mu = (s - s_d)/\beta_\mu c$, yielding

$$g_{t_\mu}(t_\mu)|_{s_d} = g_{t_\mu|p_\pi}(t_\mu) \times g_{p_\pi}(p_\pi)|_{s=s_d}$$

(1)

- Taking into account the pion density at $s$ one gets

$$g_{t_\mu}(t_\mu) = g_{t_\mu|p_\pi}(t_\mu) \times g_{p_\pi}(p_\pi)|_{s_d} \times g_s|p_\pi(s, p_\pi)$$

(2)

- A particle population can be reconstructed, left Fig. below.

Mathematica functions have been built:

- TimeDistribution[$s, E_{\pi,min}, E_{\pi,max}, Options$],
- EnergyDistribution[$s, E_{\pi,min}, E_{\pi,max}, Options$],
- LongitudinalDistribution[$s, E_{\pi,min}, E_{\pi,max}, Options$].
8 Next step: transverse phase space

8.1 Step 1: decay angle

The density of \( \text{lab} \) decay angle \( \theta_\mu \) is \( p_\pi \)-conditional and satisfies

\[
g_{\theta_\mu|p_\pi}(\theta_\mu) = \sum_{i=1,2} f(\theta_{\mu,i}^*) \left| d\theta_\mu^*/d\theta_\mu \right|_i = \frac{\gamma^2}{2} \tan \theta_\mu \sum_{i=1,2} (-1)^{i+1} \frac{\beta_\pi/\beta_\mu^* + \cos \theta_{\mu,i}^*}{1 + \beta_\pi/\beta_\mu^* \cos \theta_{\mu,i}^*}^3
\]

- with \( \theta_{\mu,i}^* \) (i=1,2) the two roots of \( \tan \theta_\mu = \frac{1}{\gamma \pi} \frac{\sin \theta_{\mu,i}^*}{\beta_\pi/\beta_\mu^* + \cos \theta_{\mu,i}^*} \)

Limit decay angles values satisfy \( \hat{\theta}_\mu = \pm \arctg \left( p_\mu^* m_\pi/ \sqrt{\left( p_\pi^2 m_\mu^2 - p_\mu^2 m_\pi^2 \right)} \right) \) as expected.
8.2 Step 2: get the “target effect” (transverse phase space)

The projection of $\theta_\mu$ onto $x$ and $x'$ or onto $z$ and $z'$ can be derived from

$$
\frac{\vec{\beta}_\mu}{|\beta_\mu|} = \begin{bmatrix}
\cos x_\pi' & -\sin x_\pi' & 0 \\
\sin x_\pi' & \cos x_\pi' & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
\cos z_\pi' & 0 & -\sin z_\pi' \\
0 & 1 & 0 \\
\sin z_\pi' & 0 & \cos z_\pi' \\
\end{bmatrix} \begin{bmatrix}
\cos \theta_\mu & 0 & -\sin \theta_\mu \\
\sin \theta_\mu \cos \phi_\mu & 1 & 0 \\
\sin \theta_\mu \sin \phi_\mu & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
x' \\
y' \\
z' \\
\end{bmatrix}
$$

(3)

We therefore know the change of variables $(\theta_\mu, \phi_\mu) \to (x, x')$, and $(\theta_\mu, \phi_\mu) \to (z, z')$.

From what we can expect to draw the densities below, as obtained from Monte Carlo simulations:

$x - x'$ muon phase space, s=40 meters downstream of production target.
The parent pion bunch is on-axis, zero transverse size, $p_\pi = 250$ MeV/c.

Eps/pi, Beta, Alpha: 7.2685E-02 92.9 -2.98 (MKSA)
in progress...
9 Conclusions

1/ Nothing precludes replacing the uniform pion momentum distribution $1_{\Delta p_{\pi}}(p_{\pi})$ used here for illustration, by more realistic ones, which will be undertaken in the frame of the decay channel studies.

2/ Next to come: second order transport of these densities through AG structures.