Chapter 19

Realistic Transfer Maps for Curved Beam-Line Elements

19.1 Introduction

Surface methods based on the use of cylinders are appropriate for straight beam-line elements or for bent elements with small sagitta. However, cylinders cannot be employed for elements with large sagitta, such as dipoles, where no straight cylinder would fit within the aperture. For such cases more complicated surfaces are required. For example, Figure 1.1 shows a bent box with straight ends. For use in the dipole case, the bent part of the box would lie within the body of the dipole, and the straight ends would enclose the fringe-field regions.

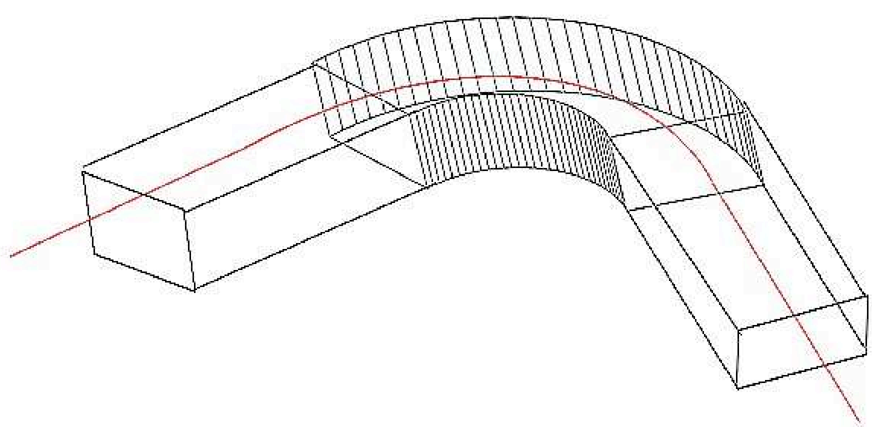

Figure 19.1.1: A bent box with straight ends.

But now there is a complication: The cylinder methods succeeded because Laplace's equation is separable in circular, elliptical, and rectangular cylinder coordinates. Consequently, we were able to find a kernel that related the interior vector potential to the normal component of the surface magnetic field. However, there is no bent coordinate system with straight ends for which Laplace's equation is separable.

This problem can in principle be overcome if both the normal component of the magnetic field and the scalar potential for the magnetic field are known on the surface. Such data are in fact provided on a mesh by some 3-dimensional field solvers, and these data can be interpolated onto the surface.

Let V be some volume in three-dimensional space bounded by a surface S. Suppose that the magnetic field $\boldsymbol{B}(\boldsymbol{r})$ is source free when $\boldsymbol{r}$ is within V. That is, for $\boldsymbol{r}$ within V, $\boldsymbol{B}(\boldsymbol{r})$ satisfies the requirements

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0, \tag{19.1.1}$$

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = 0. \tag{19.1.2}$$

This will be the case for the magnetic field in an evacuated beam pipe. For a Hamiltonian treatment of trajectories, we need a vector potential $\boldsymbol{A}(\boldsymbol{r})$ such that

$$\boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}). \tag{19.1.3}$$

Let $\boldsymbol{n}'(\boldsymbol{r}')$ be the outward normal to S at the point $\boldsymbol{r}' \in S$. Then the normal component of $\boldsymbol{B}$ on S is given by the definition

$$B_n(\boldsymbol{r}') = \boldsymbol{n}'(\boldsymbol{r}') \cdot \boldsymbol{B}(\boldsymbol{r}'). \tag{19.1.4}$$

Also, let $\psi(\boldsymbol{r}')$ be the value of the magnetic scalar potential at the point $\boldsymbol{r}' \in S$. It satisfies the relation

$$\boldsymbol{B}(\boldsymbol{r}') = \nabla' \psi(\boldsymbol{r}'). \tag{19.1.5}$$

Then, with the aid of the vector potential for Dirac magnetic monopoles and Helmholtz's theorem, it can be shown that a suitable interior vector potential $\boldsymbol{A}(\boldsymbol{r})$ for $\boldsymbol{r}$ within V is given by the relation

$$\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{A}^n(\boldsymbol{r}) + \boldsymbol{A}^t(\boldsymbol{r}) \tag{19.1.6}$$

with

$$\boldsymbol{A}^n(\boldsymbol{r}) = \int_S dS' \; B_n(\boldsymbol{r}')\boldsymbol{G}^n(\boldsymbol{r}, \boldsymbol{r}') \tag{19.1.7}$$

and

$$\boldsymbol{A}^t(\boldsymbol{r}) = \int_S dS' \; \psi(\boldsymbol{r}')\boldsymbol{G}^t(\boldsymbol{r}, \boldsymbol{r}'). \tag{19.1.8}$$

Moreover, the constituents of $\boldsymbol{A}(\boldsymbol{r})$, and hence $\boldsymbol{A}(\boldsymbol{r})$ itself, satisfy the Coulomb gauge condition,

$$\nabla \cdot \boldsymbol{A}^n(\boldsymbol{r}) = \nabla \cdot \boldsymbol{A}^t(\boldsymbol{r}) = \nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = 0. \tag{19.1.9}$$

Here the kernels $\boldsymbol{G}^n$ and $\boldsymbol{G}^t$ are given by the relations

$$\boldsymbol{G}^n(\boldsymbol{r}, \boldsymbol{r}') = \{\boldsymbol{n}'(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')\}/\{4\pi|\boldsymbol{r} - \boldsymbol{r}'|[|\boldsymbol{r} - \boldsymbol{r}'| - \boldsymbol{n}'(\boldsymbol{r}') \cdot (\boldsymbol{r} - \boldsymbol{r}')]\}, \tag{19.1.10}$$

$$\boldsymbol{G}^t(\boldsymbol{r}, \boldsymbol{r}') = [\boldsymbol{n}'(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')]/[4\pi|\boldsymbol{r} - \boldsymbol{r}'|^3]. \tag{19.1.11}$$

A detailed exposition of this method, including expected accuracy and insensitivity to noise in the surface data, is the subject of this chapter. Thus, taken together, Chapters 13 through 18 and this chapter are intended to provide an extensive description of, and associated robust numerical algorithms for, the computation of transfer maps, including all fringe-field and higher-order multipole effects, for realistic beam-line elements having arbitrary geometry.

Section 19.2 describes the the mathematical tools required to treat general geometries. These tools are Dirac's magnetic monopole vector potential and Helmloltz's theorem. Section 19.3 derives the relations (1.3) through (1.10) and describes the properties of the kernels (1.9) and (1.10). The remaining sections provide a numerical benchmark, study smoothing and insensitivity to errors, and apply the method to a storage-ring dipole.

Before continuing on, we pause to advertise some of the virtues of what can be achieved with the use of general surface methods.

- The constituents $\boldsymbol{A}^n(\boldsymbol{r})$ and $\boldsymbol{A}^t(\boldsymbol{r})$ of $\boldsymbol{A}(\boldsymbol{r})$, and hence $\boldsymbol{A}(\boldsymbol{r})$ itself, are analytic functions of $\boldsymbol{r}$ for $\boldsymbol{r}$ within V, even when there are errors in the surface fields B_n and ψ, and no matter how poorly the integrals (1.7) and (1.8) are evaluated.

- The Maxwell equations for $\boldsymbol{B}(\boldsymbol{r})$, and the Coulomb gauge condition for $\boldsymbol{A}(\boldsymbol{r})$ and its constituents, are satisfied exactly even when there are errors in the surface fields B_n and ψ, and no matter how poorly the integrals (1.7) and (1.8) are evaluated.

- The kernels $\boldsymbol{G}^n$ and $\boldsymbol{G}^t$ are smoothing. Consequently, the $\boldsymbol{A}(\boldsymbol{r})$ given by (1.6) through (1.8) is relatively insensitive to noise in the surface fields B_n and ψ.

We hasten to add that the first two items above should not be taken to mean that there is no need to take care to evaluate integrals well. They just indicate that the worst disasters have been avoided. Subsequently we will learn that the kernels $\boldsymbol{G}^n$ and $\boldsymbol{G}^t$, and their $\boldsymbol{r}$ derivatives, can be strongly peaked in $\boldsymbol{r}'$ when $\boldsymbol{r}$ is near S. To obtain accurate results, this behavior of the kernels must be taken into account when integrating, with respect to $\boldsymbol{r}'$, over the surface S.

19.2 Mathematical Tools

19.2.1 Electric Dirac Strings

In this subsection we will motivate the subject of magnetic Dirac strings by treating the simpler electric case. Suppose $\boldsymbol{E}(\boldsymbol{r})$ is a vector field that obeys the equations

$$\nabla \times \boldsymbol{E} = 0, \tag{19.2.1}$$

$$\nabla \cdot \boldsymbol{E} = \rho. \tag{19.2.2}$$

From (2.1) we know there is a scalar potential ϕ such that

$$\boldsymbol{E} = -\nabla\phi, \tag{19.2.3}$$

and from (2.2) it follows that

$$\nabla^2\phi = -\rho. \tag{19.2.4}$$

Introduce the notation

$$|\boldsymbol{r} - \boldsymbol{r}'| = ||\boldsymbol{r} - \boldsymbol{r}'|| = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}. \tag{19.2.5}$$

Consider the function $1/|\boldsymbol{r}-\boldsymbol{r}'|$. It satisfies the relation

$$\nabla^2[1/|\boldsymbol{r}-\boldsymbol{r}'|] = -4\pi\delta_3(\boldsymbol{r}-\boldsymbol{r}') \tag{19.2.6}$$

where the indicated derivatives are to be taken with respect to the components of $\boldsymbol{r}$. Assuming that $\rho(\boldsymbol{r})$ falls off sufficiently rapidly at infinity, it follows that a solution to (2.4) is given by the relation

$$\phi(\boldsymbol{r}) = [1/(4\pi)]\int d^3\boldsymbol{r}'\rho(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|. \tag{19.2.7}$$

Moreover, (2.7) is the unique solution that vanishes at infinity.

For our discussion we will need some knowledge of low-order (spherical) multipole expansions, which we review briefly here. Suppose that the charge distribution ρ is nonzero only in some volume V surrounding the point $\boldsymbol{r}_d$. (Here the subscript d stands for *distribution* and will later stand for *dipole*.) Then (2.7) becomes

$$\phi(\boldsymbol{r}) = [1/(4\pi)]\int_V d^3\boldsymbol{r}'\rho(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|. \tag{19.2.8}$$

Suppose also that $\boldsymbol{r}$ lies outside V so that the denominator in (2.8) never vanishes. Make the change of variables

$$\boldsymbol{r}' = \boldsymbol{r}_d + \boldsymbol{\xi} \tag{19.2.9}$$

so that (2.8) becomes

$$\phi(\boldsymbol{r}) = [1/(4\pi)]\int_{V_0} d^3\boldsymbol{\xi}\ \rho(\boldsymbol{r}_d+\boldsymbol{\xi})/|(\boldsymbol{r}-\boldsymbol{r}_d)-\boldsymbol{\xi}| \tag{19.2.10}$$

where V_0 is a volume surrounding the origin. Under the assumption that $\boldsymbol{r} \notin V$, the denominator factor in (2.10) can be expanded as a power series in the components of $\boldsymbol{\xi}$,

$$1/|(\boldsymbol{r}-\boldsymbol{r}_d)-\boldsymbol{\xi}| = [1/|\boldsymbol{r}-\boldsymbol{r}_d|][1+\boldsymbol{\xi}\cdot(\boldsymbol{r}-\boldsymbol{r}_d)/|\boldsymbol{r}-\boldsymbol{r}_d|^2 + O(\xi^2)]. \tag{19.2.11}$$

Put this expansion into the integral (2.10) to yield the result

$$\begin{aligned}\phi(\boldsymbol{r}) \ &= \ [1/(4\pi)][1/|\boldsymbol{r}-\boldsymbol{r}_d|]\int_{V_0} d^3\boldsymbol{\xi}\ \rho(\boldsymbol{r}_d+\boldsymbol{\xi}) \\ &\quad +[1/(4\pi)][1/|\boldsymbol{r}-\boldsymbol{r}_d|^3](\boldsymbol{r}-\boldsymbol{r}_d)\cdot\int_{V_0} d^3\boldsymbol{\xi}\ \boldsymbol{\xi}\ \rho(\boldsymbol{r}_d+\boldsymbol{\xi}) + O(\xi^2).\end{aligned} \tag{19.2.12}$$

The integrals in (2.12) can be manipulated to bring them to the forms

$$\int_{V_0} d^3\boldsymbol{\xi}\ \rho(\boldsymbol{r}_d+\boldsymbol{\xi}) = \int_V d^3\boldsymbol{r}'\ \rho(\boldsymbol{r}') = Q, \tag{19.2.13}$$

$$\int_{V_0} d^3\boldsymbol{\xi}\ \boldsymbol{\xi}\ \rho(\boldsymbol{r}_d+\boldsymbol{\xi}) = \int_V d^3\boldsymbol{r}'\ (\boldsymbol{r}'-\boldsymbol{r}_d)\ \rho(\boldsymbol{r}') = \boldsymbol{p}_d. \tag{19.2.14}$$

Here Q, the total charge in V, is the monopole moment, and $\boldsymbol{p}_d$ is the dipole moment (with respect to the point $\boldsymbol{r}_d$) of the charge distribution in V. Thus, we find that

$$\phi(\boldsymbol{r}) = [Q/(4\pi)][1/|\boldsymbol{r}-\boldsymbol{r}_d|] + [1/(4\pi)][\boldsymbol{p}_d\cdot(\boldsymbol{r}-\boldsymbol{r}_d)]/|\boldsymbol{r}-\boldsymbol{r}_d|^3 + O(\xi^2). \tag{19.2.15}$$

That is, the potential arising from a charge distribution, at a point $\boldsymbol{r}$ outside the distribution, is a sum of monopole, dipole, and higher-order multipole contributions.

We recall that the prototypical example of a dipole consists of two opposite charges $\pm q$ separated by a distance 2ϵ in the limit that $\epsilon \to 0$ and $q \to \infty$ in such a way that the product $2q\epsilon$ remains constant. For example, suppose a charge $+q$ is placed at the location $\boldsymbol{r}_d + \boldsymbol{\epsilon}$ and a charge $-q$ is placed at the location $\boldsymbol{r}_d - \boldsymbol{\epsilon}$. Then we find that the potential due to this two-charge combination is given by the relation

$$\phi(\boldsymbol{r}, \boldsymbol{r}_d) = [1/(4\pi)][q/|\boldsymbol{r} - (\boldsymbol{r}_d + \boldsymbol{\epsilon})| - q/|\boldsymbol{r} - (\boldsymbol{r}_d - \boldsymbol{\epsilon})|]. \tag{19.2.16}$$

Expansion of (2.16) in powers of $\boldsymbol{\epsilon}$ gives the result

$$\phi(\boldsymbol{r}, \boldsymbol{r}_d) = [1/(4\pi)](2q\boldsymbol{\epsilon}) \cdot (\boldsymbol{r} - \boldsymbol{r}_d)/|\boldsymbol{r} - \boldsymbol{r}_d|^3 + O(q\epsilon^2). \tag{19.2.17}$$

Now let $\boldsymbol{\epsilon} \to 0$ and $q \to \infty$ in such a way that

$$2q\boldsymbol{\epsilon} \to \boldsymbol{p}_d. \tag{19.2.18}$$

In this limit (2.17) becomes

$$\phi_d(\boldsymbol{r}, \boldsymbol{r}_d) = [1/(4\pi)][\boldsymbol{p}_d \cdot (\boldsymbol{r} - \boldsymbol{r}_d)]/|\boldsymbol{r} - \boldsymbol{r}_d|^3, \tag{19.2.19}$$

in agreement with the second term in (2.15). We note, with the convention $q > 0$, that the dipole moment vector $\boldsymbol{p}_d$ points from the location of $-q$ to the location of $+q$.

We also note, for future use, that the field $\boldsymbol{E}_d(\boldsymbol{r}, \boldsymbol{r}_d)$ at the point $\boldsymbol{r}$ arising from a dipole at the point $\boldsymbol{r}_d$ (with $\boldsymbol{r} \neq \boldsymbol{r}_d$) is given by the relation

$$\begin{aligned} \boldsymbol{E}_d(\boldsymbol{r}, \boldsymbol{r}_d) &= -\nabla \phi_d(\boldsymbol{r}, \boldsymbol{r}_d) \\ &= -[1/(4\pi)][\boldsymbol{p}_d/|\boldsymbol{r} - \boldsymbol{r}_d|^3] + [3/(4\pi)](\boldsymbol{r} - \boldsymbol{r}_d)[\boldsymbol{p}_d \cdot (\boldsymbol{r} - \boldsymbol{r}_d)]/|\boldsymbol{r} - \boldsymbol{r}_d|^5. \end{aligned} \tag{19.2.20}$$

We will now use the expression for the potential of a dipole, namely (2.19), to carry out an instructive construction and calculation. Suppose $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ are the locations of two points A and B. Imagine these two points to be joined by a line (path, *string*) L starting at $\boldsymbol{r}_A$ and ending at $\boldsymbol{r}_B$. See Figure 2.1. Divide the path into N segments, each of length Δs, and place a dipole of magnitude $g\Delta s$ at the center of each segment with the dipole moment vector pointing along the path at each point. Here g is some constant. Thus, the dipole moment $\Delta \boldsymbol{p}_d$ of each segment is given by the expression

$$\Delta \boldsymbol{p}_d = g\Delta s(\Delta \boldsymbol{r}/|\Delta \boldsymbol{r}|) = g\Delta \boldsymbol{r} \tag{19.2.21}$$

since $|\Delta \boldsymbol{r}| = \Delta s$. Let us compute the potential $\phi_s(\boldsymbol{r})$ produced by this *string* of dipoles. It will be the sum of the potentials of the individual dipoles. In the limit $\Delta s \to 0$ and $N \to \infty$ it is given by the integral

$$\begin{aligned} \phi_s(\boldsymbol{r}) &= [1/(4\pi)] \int_L d\boldsymbol{p}_d \cdot (\boldsymbol{r} - \boldsymbol{r}_d)/|\boldsymbol{r} - \boldsymbol{r}_d|^3 \\ &= [g/(4\pi)] \int_{\boldsymbol{r}_A}^{\boldsymbol{r}_B} d\boldsymbol{r}_d \cdot (\boldsymbol{r} - \boldsymbol{r}_d)/|\boldsymbol{r} - \boldsymbol{r}_d|^3. \end{aligned} \tag{19.2.22}$$

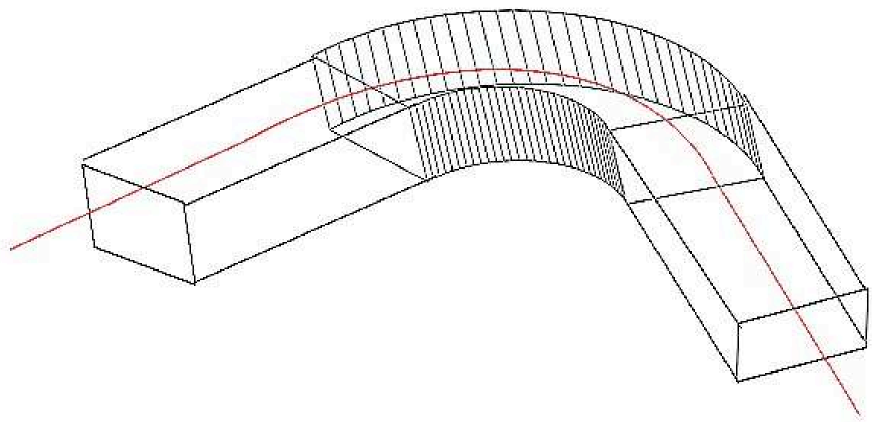

Figure 19.2.1: (Place Holder) A path L from the point A to the point B. Dipoles are laid out and aligned along the path to form a string.

Can the integral (2.22) be evaluated? Recall the identity

$$\nabla^d(1/|\boldsymbol{r}-\boldsymbol{r}_d|) = (\boldsymbol{r}-\boldsymbol{r}_d)/|\boldsymbol{r}-\boldsymbol{r}_d|^3 \tag{19.2.23}$$

where ∇^d denotes differentiation with respect to the components of $\boldsymbol{r}_d$. This identity may be employed in (2.22) to yield the result

$$\begin{aligned}\phi_s(\boldsymbol{r}) &= [g/(4\pi)]\int_{\boldsymbol{r}_A}^{\boldsymbol{r}_B} d\boldsymbol{r}_d\cdot(\boldsymbol{r}-\boldsymbol{r}_d)/|\boldsymbol{r}-\boldsymbol{r}_d|^3\\ &= [g/(4\pi)]\int_{\boldsymbol{r}_A}^{\boldsymbol{r}_B} d\boldsymbol{r}_d\cdot[\nabla^d(1/|\boldsymbol{r}-\boldsymbol{r}_d|)]\\ &= [g/(4\pi)]\{[(1/|\boldsymbol{r}-\boldsymbol{r}_B|)]-[(1/|\boldsymbol{r}-\boldsymbol{r}_A|)]\}.\end{aligned} \tag{19.2.24}$$

We see that the potential $\phi_s(\boldsymbol{r})$ resulting from a string of dipoles is the same as the potential produced by a charge $-g$ located at $\boldsymbol{r}_A$ and a charge $+g$ located at $\boldsymbol{r}_B$. This mathematically derived result is also intuitive because we expect, for a string of dipoles arrayed head-to-tail, that adjacent head-tail pairs would cancel so all that would be left would be the negative initial tail and the final positive head.

Note that, as it stands, (2.22) is undefined for points $\boldsymbol{r}\in L$. However, since the integrand in (2.22) is a perfect differential, see (2.23), the path can be deformed at will to avoid any possible vanishings of the denominator in (2.22) without changing the value of the integral. Indeed, (2.24) shows that $\phi_s(\boldsymbol{r})$ depends only on the endpoints of the path, and is otherwise path independent.

19.2.2 Magnetic Dirac Strings

In analogy to the work of the previous subsection, this subsection will describe calculations for the complementary case of a vector field $\boldsymbol{B}(\boldsymbol{r})$ that obeys the equations

$$\nabla\times\boldsymbol{B} = \boldsymbol{J}, \tag{19.2.25}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{19.2.26}$$

Note, in order for (2.25) to make sense, we must require that

$$\nabla \cdot \boldsymbol{J} = \nabla \cdot (\nabla \times \boldsymbol{B}) = 0. \tag{19.2.27}$$

(Recall that the divergence of a curl vanishes.)

In the case of (2.25) and (2.26) it is often assumed that there is a vector potential $\boldsymbol{A}(\boldsymbol{r})$ such that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{19.2.28}$$

because (2.26) will then be satisfied automatically. Let us verify that this Ansatz is possible by construction. Substitution of (2.28) into (2.25) yields the hypothesis

$$\nabla \times (\nabla \times \boldsymbol{A}) = \boldsymbol{J}. \tag{19.2.29}$$

Recall the vector identity

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \tag{19.2.30}$$

where here it is essential that Cartesian components be employed. Let us make the further Coulomb gauge assumption

$$\nabla \cdot \boldsymbol{A} = 0. \tag{19.2.31}$$

In this circumstance (2.30) becomes

$$\nabla^2 \boldsymbol{A} = -\boldsymbol{J}. \tag{19.2.32}$$

Because of (2.6), equation (2.32) has the immediate solution

$$\boldsymbol{A}(\boldsymbol{r}) = [1/(4\pi)] \int d^3\boldsymbol{r}' \, \boldsymbol{J}(\boldsymbol{r}')/|\boldsymbol{r} - \boldsymbol{r}'|. \tag{19.2.33}$$

Moreover, (2.33) is the unique solution that vanishes at infinity. But wait, we must also verify that (2.33) also satisfies (2.31). It does, as you will have the pleasure of showing in Exercise 2.4.

Next suppose that the current distribution $\boldsymbol{J}$ is nonzero only in some volume V surrounding the point $\boldsymbol{r}_d$. Then (2.33) becomes

$$\boldsymbol{A}(\boldsymbol{r}) = [1/(4\pi)] \int_V d^3\boldsymbol{r}' \, \boldsymbol{J}(\boldsymbol{r}')/|\boldsymbol{r} - \boldsymbol{r}'|. \tag{19.2.34}$$

Suppose also that $\boldsymbol{r}$ lies outside V so that the denominator in (2.34) never vanishes. Make the change of variables (2.9) so that (2.34) can be rewritten in the form

$$\boldsymbol{A}(\boldsymbol{r}) = [1/(4\pi)] \int_{V_0} d^3\boldsymbol{\xi} \, \boldsymbol{J}(\boldsymbol{r}_d + \boldsymbol{\xi})/|(\boldsymbol{r} - \boldsymbol{r}_d) - \boldsymbol{\xi}| \tag{19.2.35}$$

where V_0 is a volume surrounding the origin. As before, make the expansion (2.11) so that (2.35) can be written in the form

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}) &= [1/(4\pi)][1/|\boldsymbol{r}-\boldsymbol{r}_d|]\int_{V_0} d^3\boldsymbol{\xi}\ \boldsymbol{J}(\boldsymbol{r}_d+\boldsymbol{\xi}) \\ &\quad +[1/(4\pi)][1/|\boldsymbol{r}-\boldsymbol{r}_d|^3]\int_{V_0} d^3\boldsymbol{\xi}\ [(\boldsymbol{r}-\boldsymbol{r}_d)\cdot\boldsymbol{\xi}]\ \boldsymbol{J}(\boldsymbol{r}_d+\boldsymbol{\xi})+O(\xi^2). \end{aligned} \tag{19.2.36}$$

The integrals in (2.36) can again be manipulated to bring them to more convenient forms. For the first integral we find that

$$\int_{V_0} d^3\boldsymbol{\xi}\ \boldsymbol{J}(\boldsymbol{r}_d+\boldsymbol{\xi}) = \int_V d^3\boldsymbol{r}'\ \boldsymbol{J}(\boldsymbol{r}')=0. \tag{19.2.37}$$

Here use has been made of (2.27). See Exercise 2.5. The second integral can be brought to the form

$$\begin{aligned} \int_{V_0} d^3\boldsymbol{\xi}\ [(\boldsymbol{r}-\boldsymbol{r}_d)\cdot\boldsymbol{\xi}]\ \boldsymbol{J}(\boldsymbol{r}_d+\boldsymbol{\xi}) &= \int_V d^3\boldsymbol{r}'\ [(\boldsymbol{r}-\boldsymbol{r}_d)\cdot(\boldsymbol{r}'-\boldsymbol{r}_d)]\ \boldsymbol{J}(\boldsymbol{r}') \\ &= \boldsymbol{m}_d\times(\boldsymbol{r}-\boldsymbol{r}_d). \end{aligned} \tag{19.2.38}$$

Here use has again been made of (2.27), and $\boldsymbol{m}_d$ is the magnetic dipole moment defined by the integral

$$\boldsymbol{m}_d=(1/2)\int_V d^3\boldsymbol{r}'\ [(\boldsymbol{r}'-\boldsymbol{r}_d)\times\boldsymbol{J}(\boldsymbol{r}')]. \tag{19.2.39}$$

See Exercise 2.6. Thus, we find that

$$\boldsymbol{A}(\boldsymbol{r})=\boldsymbol{A}_d(\boldsymbol{r},\boldsymbol{r}_d)+O(\xi^2) \tag{19.2.40}$$

where

$$\boldsymbol{A}_d(\boldsymbol{r},\boldsymbol{r}_d)=[1/(4\pi)][\boldsymbol{m}_d\times(\boldsymbol{r}-\boldsymbol{r}_d)]/|\boldsymbol{r}-\boldsymbol{r}_d|^3. \tag{19.2.41}$$

We see that the vector potential arising from a current distribution, at a point $\boldsymbol{r}$ outside the distribution, is a sum of dipole and higher order multipole contributions. Unlike the electric case, there is no monopole contribution. We also remark that $\boldsymbol{A}(\boldsymbol{r},\boldsymbol{r}_d)$ satisfies the Coulomb gauge condition (2.31),

$$\nabla\cdot\boldsymbol{A}(\boldsymbol{r},\boldsymbol{r}_d)=0. \tag{19.2.42}$$

See Exercise 2.7.

We recall that the prototypical example of a magnetic dipole consists of a small circular and planar ring of radius R, surrounding an area A and carrying a current I, in the limit that $A\to 0$ and $I\to\infty$ in such a way that the product AI remains constant. For example, suppose the ring is placed in the x,y plane and centered around the origin. Suppose also that the current I circulates in the counterclockwise direction when viewed from above (looking down from positive z toward the origin). Then we find that (2.39) takes the form

$$\boldsymbol{m}_d=(1/2)\int_V d^3\boldsymbol{r}'\ [\boldsymbol{r}'\times\boldsymbol{J}(\boldsymbol{r}')]=AI\boldsymbol{e}_z. \tag{19.2.43}$$

For the field $\boldsymbol{B}_d(\boldsymbol{r}, \boldsymbol{r}_d)$ at the point $\boldsymbol{r}$ arising from a magnetic dipole at the point $\boldsymbol{r}_d$ (with $\boldsymbol{r} \neq \boldsymbol{r}_d$) we find the result

$$\begin{aligned} \boldsymbol{B}_d(\boldsymbol{r}, \boldsymbol{r}_d) &= \nabla \times \boldsymbol{A}_d(\boldsymbol{r}, \boldsymbol{r}_d) \\ &= -[1/(4\pi)][\boldsymbol{m}_d/|\boldsymbol{r} - \boldsymbol{r}_d|^3] + [3/(4\pi)](\boldsymbol{r} - \boldsymbol{r}_d)[\boldsymbol{m}_d \cdot (\boldsymbol{r} - \boldsymbol{r}_d)]/|\boldsymbol{r} - \boldsymbol{r}_d|^5. \end{aligned} \tag{19.2.44}$$

Note that (2.20) and (2.44) agree if $\boldsymbol{p}_d = \boldsymbol{m}_d$. Thus, we have the key relation

$$-\nabla\phi_d(\boldsymbol{r}, \boldsymbol{r}_d) = \nabla \times \boldsymbol{A}_d(\boldsymbol{r}, \boldsymbol{r}_d) \text{ when } \boldsymbol{p}_d = \boldsymbol{m}_d \text{ and } \boldsymbol{r} \neq \boldsymbol{r}_d. \tag{19.2.45}$$

In analogy to what was done in the previous subsection for a string of electric dipoles, let us compute the vector potential $\boldsymbol{A}_s(\boldsymbol{r})$ arising from a string of magnetic dipoles. Again we will initially divide the path into N equal segments, and the magnetic dipole moment of each segment will be given by the relation

$$\Delta\boldsymbol{m}_d = g\Delta s(\Delta\boldsymbol{r}/|\Delta\boldsymbol{r}|) = g\Delta\boldsymbol{r}. \tag{19.2.46}$$

In the limit $\Delta s \to 0$ and $N \to \infty$ the vector potential due to the string is given by the integral

$$\begin{aligned} \boldsymbol{A}_s(\boldsymbol{r}) &= [1/(4\pi)] \int_L d\boldsymbol{m}_d \times (\boldsymbol{r} - \boldsymbol{r}_d)/|\boldsymbol{r} - \boldsymbol{r}_d|^3 \\ &= [g/(4\pi)] \int_{\boldsymbol{r}_A}^{\boldsymbol{r}_B} d\boldsymbol{r}_d \times (\boldsymbol{r} - \boldsymbol{r}_d)/|\boldsymbol{r} - \boldsymbol{r}_d|^3. \end{aligned} \tag{19.2.47}$$

Recall (2.41). Note that, as it stands, (2.47) is undefined for points $\boldsymbol{r} \in L$. As before, the path can be deformed to avoid any possible vanishings of the denominator. However, unlike the electric case and as will soon be seen, so doing changes the value of $\boldsymbol{A}_s(\boldsymbol{r})$. We also note that the current distribution associated with a string of magnetic dipoles (all aligned along the string) is that of an infinitesimally thin solenoid bent into the shape of the string.

What is the nature of the magnetic field $\boldsymbol{B}_s(\boldsymbol{r})$ given by

$$\boldsymbol{B}_s(\boldsymbol{r}) = \nabla \times \boldsymbol{A}_s(\boldsymbol{r})? \tag{19.2.48}$$

We claim, for $\boldsymbol{r} \notin L$, that

$$\nabla \times \boldsymbol{A}_s(\boldsymbol{r}) = -\nabla\phi_s(\boldsymbol{r}). \tag{19.2.49}$$

We will prove this assertion shortly. Assuming it is true, the right side of (2.48) can be evaluated easily using (2.49). In view of (2.24), there is the relation

$$-\nabla\phi_s(\boldsymbol{r}) = [g/(4\pi)][(\boldsymbol{r} - \boldsymbol{r}_B)/|\boldsymbol{r} - \boldsymbol{r}_B|^3] - [g/(4\pi)][(\boldsymbol{r} - \boldsymbol{r}_A)/|\boldsymbol{r} - \boldsymbol{r}_A|^3]. \tag{19.2.50}$$

It follows that

$$\boldsymbol{B}_s(\boldsymbol{r}) = [g/(4\pi)][(\boldsymbol{r} - \boldsymbol{r}_B)/|\boldsymbol{r} - \boldsymbol{r}_B|^3] - [g/(4\pi)][(\boldsymbol{r} - \boldsymbol{r}_A)/|\boldsymbol{r} - \boldsymbol{r}_A|^3]. \tag{19.2.51}$$

We see that the field $\boldsymbol{B}_s(\boldsymbol{r})$ is that produced by two magnetic *monopoles*, one located at $\boldsymbol{r}_B$ with strength g, and a second located at $\boldsymbol{r}_A$ with strength $-g$.

At this juncture two comments are in order. First, the $\boldsymbol{B}_s(\boldsymbol{r})$ given by (2.51) evidently *is not* divergence free at the points $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$. But the $\boldsymbol{B}_s(\boldsymbol{r})$ given by (2.48) is a curl, and we again recall the theorem that a curl *is* divergence free. The resolution to this apparent paradox is that $\boldsymbol{A}_s(\boldsymbol{r})$ is singular for $\boldsymbol{r} \in L$, and every neighborhood of the points $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ contains such singular points, and therefore the conditions for the theorem are not met. Correspondingly, (2.51) holds only for points $\boldsymbol{r} \notin L$.

The second comment is equally subtle. Suppose two different strings s and s' (but with the same endpoints) are used to compute $\boldsymbol{B}_s(\boldsymbol{r})$ and $\boldsymbol{B}_{s'}(\boldsymbol{r})$. Then, according to (2.51), these fields should agree except possibly at the points for which $\boldsymbol{r} \in L$ and/or $\boldsymbol{r} \in L'$. Thus, we have the relation

$$\nabla \times [\boldsymbol{A}_s(\boldsymbol{r}) - \boldsymbol{A}_{s'}(\boldsymbol{r})] = 0 \text{ for } \boldsymbol{r} \notin L \text{ and } \boldsymbol{r} \notin L'. \tag{19.2.52}$$

Let Σ be some surface spanning the two strings s and s'. See Figure 2.2. Three-dimensional Euclidean space with the surface Σ excluded is still simply connected. It follows that there is a function $\psi_{ss'}(\boldsymbol{r})$ such that

$$\boldsymbol{A}_s(\boldsymbol{r}) - \boldsymbol{A}_{s'}(\boldsymbol{r}) = \nabla\psi_{ss'}(\boldsymbol{r}) \text{ for } \boldsymbol{r} \notin \Sigma. \tag{19.2.53}$$

That is, the vector potentials associated with two different strings (but with the same endpoints) are related by a gauge transformation. From (2.53) we see that $\psi_{ss'}(\boldsymbol{r})$ will be singular for both $\boldsymbol{r} \in L$ and $\boldsymbol{r} \in L'$. It can be shown that $\psi_{ss'}(\boldsymbol{r})$ is also harmonic,

$$\nabla^2\psi_{ss'}(\boldsymbol{r}) = 0 \text{ for } \boldsymbol{r} \notin L \text{ and } \boldsymbol{r} \notin L'. \tag{19.2.54}$$

See Exercise 2.13.

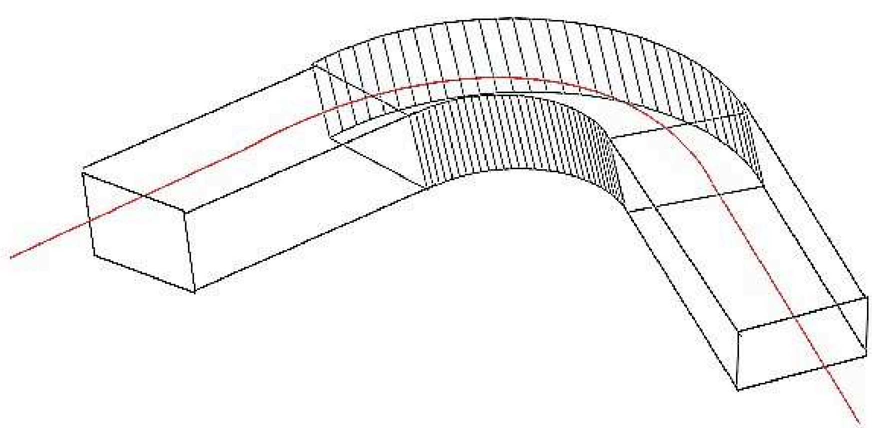

Figure 19.2.2: (Place holder.) A surface Σ spanning the two strings s and s'.

Finally, suppose we let $\boldsymbol{r}_B \to \infty$. In this limit, the first term on the right side of (2.51) vanishes, and we have the result

$$\boldsymbol{B}_s(\boldsymbol{r}) = -[g/(4\pi)][(\boldsymbol{r} - \boldsymbol{r}_A)/|\boldsymbol{r} - \boldsymbol{r}_A|^3], \tag{19.2.55}$$

which is the field of a monopole located at $\boldsymbol{r}_A$ and having strength $-g$. Correspondingly, the upper limit in the integral (2.47) is infinite, and the string s, commonly called a half-infinite *Dirac* string, extends from $\boldsymbol{r}_A$ to infinity. And the field (2.55) may be viewed as that of a *Dirac* magnetic monopole. See Exercise 2.14.

For future use, there is a special class of half-infinite strings that is particularly convenient. Let $\boldsymbol{m}$ be some unit vector. Consider the straight string (path) from $\boldsymbol{r}_A$ to infinity parameterized as

$$\boldsymbol{r}_d(\lambda) = \boldsymbol{r}_A + \lambda \boldsymbol{m} \text{ with } \lambda \in [0, \infty]. \tag{19.2.56}$$

See Figure 2.3. Then, on this path, $\boldsymbol{m}_d$ is in the direction of $\boldsymbol{m}$, and we also have the relation

$$d\boldsymbol{r}_d(\lambda) = \boldsymbol{m} d\lambda. \tag{19.2.57}$$

For this class of strings the integral (2.47) can be evaluated analytically. We begin by rewriting (2.47) in the form

$$\boldsymbol{A}_s(\boldsymbol{r}; \boldsymbol{r}_A, \boldsymbol{m}) = [g/(4\pi)] \int_0^\infty d\lambda\, \boldsymbol{m} \times [\boldsymbol{r} - \boldsymbol{r}_d(\lambda)]/|\boldsymbol{r} - \boldsymbol{r}_d(\lambda)|^3. \tag{19.2.58}$$

From (2.56) we see that

$$\boldsymbol{r} - \boldsymbol{r}_d(\lambda) = \boldsymbol{r} - \boldsymbol{r}_A - \lambda \boldsymbol{m} \tag{19.2.59}$$

and therefore

$$\boldsymbol{m} \times [\boldsymbol{r} - \boldsymbol{r}_d(\lambda)] = \boldsymbol{m} \times (\boldsymbol{r} - \boldsymbol{r}_A). \tag{19.2.60}$$

Consequently, the integral (2.58) simplifies to the form

$$\boldsymbol{A}_s(\boldsymbol{r}; \boldsymbol{r}_A, \boldsymbol{m}) = [g/(4\pi)][\boldsymbol{m} \times (\boldsymbol{r} - \boldsymbol{r}_A)] \int_0^\infty d\lambda/|\boldsymbol{r} - \boldsymbol{r}_d(\lambda)|^3. \tag{19.2.61}$$

As shown in Exercise 2.14, the integral appearing in (2.61) can be evaluated to yield the result

$$\begin{aligned} \int_0^\infty d\lambda/|\boldsymbol{r} - \boldsymbol{r}_d(\lambda)|^3 &= \int_0^\infty d\lambda/|\boldsymbol{r} - \boldsymbol{r}_A - \lambda \boldsymbol{m}|^3 \\ &= 1/\{|\boldsymbol{r} - \boldsymbol{r}_A|[|\boldsymbol{r} - \boldsymbol{r}_A| - \boldsymbol{m} \cdot (\boldsymbol{r} - \boldsymbol{r}_A)]\}. \end{aligned} \tag{19.2.62}$$

Therefore, $\boldsymbol{A}_s(\boldsymbol{r}; \boldsymbol{r}_A, \boldsymbol{m})$ takes the final explicit form

$$\boldsymbol{A}_s(\boldsymbol{r}; \boldsymbol{r}_A, \boldsymbol{m}) = [g/(4\pi)][\boldsymbol{m} \times (\boldsymbol{r} - \boldsymbol{r}_A)]/\{|\boldsymbol{r} - \boldsymbol{r}_A|[|\boldsymbol{r} - \boldsymbol{r}_A| - \boldsymbol{m} \cdot (\boldsymbol{r} - \boldsymbol{r}_A)]\}. \tag{19.2.63}$$

It remains to be verified that (2.49) holds. Suppose that (2.19) is written in the form

$$\phi_d(\boldsymbol{r}, \boldsymbol{r}_d; |\boldsymbol{p}_d|, \boldsymbol{n}_d) = [1/(4\pi)][|\boldsymbol{p}_d| \boldsymbol{n}_d \cdot (\boldsymbol{r} - \boldsymbol{r}_d)]/|\boldsymbol{r} - \boldsymbol{r}_d|^3 \tag{19.2.64}$$

where $\boldsymbol{n}_d$ is the unit vector in the direction of $\boldsymbol{p}_d$. Then (2.22) takes the form

$$\phi_s(\boldsymbol{r}) = \int_L \phi_d(\boldsymbol{r}, \boldsymbol{r}_d; g ds, d\boldsymbol{r}_d/|d\boldsymbol{r}_d|), \tag{19.2.65}$$

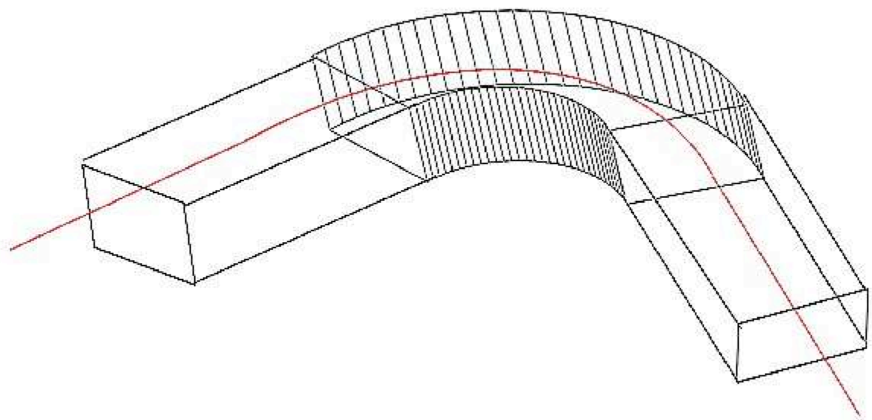

Figure 19.2.3: (Place holder.) A straight half-infinite string extending from A to infinity in the direction $\boldsymbol{m}$.

and therefore

$$-\nabla\phi_s(\boldsymbol{r}) = \int_L -\nabla\phi_d(\boldsymbol{r}, \boldsymbol{r}_d; gds, d\boldsymbol{r}_d/|d\boldsymbol{r}_d|). \tag{19.2.66}$$

Suppose also that (2.41) is written in the form

$$\boldsymbol{A}_d(\boldsymbol{r}, \boldsymbol{r}_d; |\boldsymbol{m}_d|, \boldsymbol{n}_d) = [1/(4\pi)][|\boldsymbol{m}_d|\boldsymbol{n}_d \times (\boldsymbol{r} - \boldsymbol{r}_d)]/|\boldsymbol{r} - \boldsymbol{r}_d|^3. \tag{19.2.67}$$

Then (2.47) takes the form

$$\boldsymbol{A}_s(\boldsymbol{r}) = \int_L \boldsymbol{A}_d(\boldsymbol{r}, \boldsymbol{r}_d; gds, d\boldsymbol{r}_d/|d\boldsymbol{r}_d|), \tag{19.2.68}$$

and therefore

$$\nabla \times \boldsymbol{A}_s(\boldsymbol{r}) = \int_L \nabla \times \boldsymbol{A}_d(\boldsymbol{r}, \boldsymbol{r}_d; gds, d\boldsymbol{r}_d/|d\boldsymbol{r}_d|). \tag{19.2.69}$$

Now compare the integrands on the right sides of (2.66) and (2.69). We see that they have identical arguments. Consequently, by (2.45), they are equal. It follows that the left sides of (2.66) and (2.69) are equal, and therefore (2.49) is correct.

There are still two final matters. First, (2.67) shows that $\boldsymbol{A}_s(\boldsymbol{r})$ is a superposition (integration over $\boldsymbol{r}_d$) of the $\boldsymbol{A}_d(\boldsymbol{r}, \boldsymbol{r}_d)$ and, for each $\boldsymbol{A}_d(\boldsymbol{r}, \boldsymbol{r}_d)$, the relation (2.42) holds. It follows that $\boldsymbol{A}_s(\boldsymbol{r})$ also satisfies the Coulomb gauge condition,

$$\nabla \cdot \boldsymbol{A}_s(\boldsymbol{r}) = 0. \tag{19.2.70}$$

In particular, there is the relation

$$\nabla \cdot \boldsymbol{A}_s(\boldsymbol{r}; \boldsymbol{r}_A, \boldsymbol{m}) = 0. \tag{19.2.71}$$

Second, since $\boldsymbol{A}_s(\boldsymbol{r};\boldsymbol{r}_A,\boldsymbol{m})$, is a magnetic monopole vector potential, there is the relation

$$\nabla\times\boldsymbol{A}_s(\boldsymbol{r};\boldsymbol{r}_A,\boldsymbol{m})=-[g/(4\pi)][(\boldsymbol{r}-\boldsymbol{r}_A)/|\boldsymbol{r}-\boldsymbol{r}_A|^3]=[g/(4\pi)]\nabla(1/|\boldsymbol{r}-\boldsymbol{r}_A|). \quad (19.2.72)$$

It follows that

$$\nabla\times[\nabla\times\boldsymbol{A}_s(\boldsymbol{r};\boldsymbol{r}_A,\boldsymbol{m})]=0. \quad (19.2.73)$$

The relations (2.71) and (2.73) will be of subsequent use.

19.2.3 Helmholtz Decomposition

Suppose V is some simply connected volume in 3-dimensional space bounded by a surface S, and suppose $\boldsymbol{F}(\boldsymbol{r})$ is some 3-dimensional vector field defined in V. Then, according to a theorem of *Helmholtz*, there are scalar and vector potentials $\phi(\boldsymbol{r})$ and $\boldsymbol{A}(\boldsymbol{r})$ such that

$$\boldsymbol{F}(\boldsymbol{r})=-\nabla\phi(\boldsymbol{r})+\nabla\times\boldsymbol{A}(\boldsymbol{r}) \quad (19.2.74)$$

for $\boldsymbol{r}\in V$. Moreover, $\boldsymbol{A}(\boldsymbol{r})$ will have the property

$$\nabla\cdot\boldsymbol{A}(\boldsymbol{r})=0 \quad (19.2.75)$$

for $\boldsymbol{r}\in V$. Finally, let Let $G(\boldsymbol{r},\boldsymbol{r}')$ be the function

$$G(\boldsymbol{r},\boldsymbol{r}')=1/|\boldsymbol{r}-\boldsymbol{r}'|. \quad (19.2.76)$$

Then, the scalar and vector potentials are given in terms of $\boldsymbol{F}(\boldsymbol{r})$, with $\boldsymbol{r}\in V$, by the relations

$$\phi(\boldsymbol{r})=-[1/(4\pi)]\int_S dS'\,\boldsymbol{n}'\cdot\boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')+[1/(4\pi)]\int_V d^3\boldsymbol{r}'\,G(\boldsymbol{r},\boldsymbol{r}')\nabla'\cdot\boldsymbol{F}(\boldsymbol{r}'), \quad (19.2.77)$$

$$\boldsymbol{A}(\boldsymbol{r})=-[1/(4\pi)]\int_S dS'\,[\boldsymbol{n}'\times G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')]+[1/(4\pi)]\int_V d^3\boldsymbol{r}'\,G(\boldsymbol{r},\boldsymbol{r}')\nabla'\times\boldsymbol{F}(\boldsymbol{r}'). \quad (19.2.78)$$

Here $\boldsymbol{n}'$ is the outward normal to S at the point $\boldsymbol{r}'$.

We will derive this result in stages. Before doing so, some remarks are in order. There two cases of special interest. If $\boldsymbol{F}(\boldsymbol{r})$ is globally defined and falls off at infinity as fast as $1/|\boldsymbol{r}|^2$, then we may take the surface S to infinity and find that the surface integrals vanish. This result shows that, with suitable boundary conditions (fall off) imposed at infinity, $\boldsymbol{F}(\boldsymbol{r})$ is completely specified in terms of its divergence and curl. That the operations of divergence and curl are necessary and sufficient to determine $\boldsymbol{F}(\boldsymbol{r})$ is a consequence of two things: the fact that we are working in *three* dimensions and certain properties of the Euclidean group in three dimensions. See Exercise 2.20.

The second case, of special interest for our purposes, is that for which $\boldsymbol{F}(\boldsymbol{r})$ is divergence and curl free (source free) in V,

$$\nabla\cdot\boldsymbol{F}(\boldsymbol{r})=0 \quad (19.2.79)$$

and

$$\nabla \times \boldsymbol{F}(\boldsymbol{r}) = 0 \tag{19.2.80}$$

for $\boldsymbol{r} \in V$. In this case, only the surface terms appear in (2.77) and (2.78), and we obtain the results

$$\phi(\boldsymbol{r}) = -[1/(4\pi)] \int_S dS' \, \boldsymbol{n}' \cdot \boldsymbol{F}(\boldsymbol{r}') G(\boldsymbol{r}, \boldsymbol{r}'), \tag{19.2.81}$$

$$\boldsymbol{A}(\boldsymbol{r}) = -[1/(4\pi)] \int_S dS' \, [\boldsymbol{n}' \times \boldsymbol{F}(\boldsymbol{r}')] G(\boldsymbol{r}, \boldsymbol{r}'). \tag{19.2.82}$$

We will eventually apply these results to the case of a magnetic field $\boldsymbol{B}(\boldsymbol{r})$ that is assumed to be source free within V, as in (1.1) and (1.2). We take the opportunity at this point to note that $G(\boldsymbol{r}, \boldsymbol{r}')$ as given by (2.76), and for fixed $\boldsymbol{r}'$, is an *analytic* function of the components of $\boldsymbol{r}$ for $\boldsymbol{r} \neq \boldsymbol{r}'$. It follows from the representations (2.81) and (2.82), under very mild assumptions on the surface behavior of $\boldsymbol{F}(\boldsymbol{r})$, boundedness and continuity will do, that $\phi(\boldsymbol{r})$ and $\boldsymbol{A}(\boldsymbol{r})$ are analytic functions of the components of $\boldsymbol{r}$ for $\boldsymbol{r}$ within V. Correspondingly, from (2.74), $\boldsymbol{F}(\boldsymbol{r})$ must then also be analytic for $\boldsymbol{r}$ within V.

We begin the proof of Helmholtz's theorem by noting that $G(\boldsymbol{r}, \boldsymbol{r}')$ has the properties

$$\nabla G(\boldsymbol{r}, \boldsymbol{r}') = -\nabla' G(\boldsymbol{r}, \boldsymbol{r}') = -(\boldsymbol{r} - \boldsymbol{r}')/|\boldsymbol{r} - \boldsymbol{r}'|^3, \tag{19.2.83}$$

$$\nabla^2 G(\boldsymbol{r}, \boldsymbol{r}') = (\nabla')^2 G(\boldsymbol{r}, \boldsymbol{r}') = -4\pi\delta_3(\boldsymbol{r} - \boldsymbol{r}'), \tag{19.2.84}$$

where ∇' denotes differentiation with respect to the components of $\boldsymbol{r}'$. As a result of (2.84) there is, for $\boldsymbol{r} \in V$, the identity

$$\begin{aligned}
\boldsymbol{F}(\boldsymbol{r}) &= \int_V d^3\boldsymbol{r}' \, \delta_3(\boldsymbol{r} - \boldsymbol{r}') \boldsymbol{F}(\boldsymbol{r}') \\
&= -[1/(4\pi)] \int_V d^3\boldsymbol{r}' \, \boldsymbol{F}(\boldsymbol{r}') \nabla^2 G(\boldsymbol{r}, \boldsymbol{r}') \\
&= -[1/(4\pi)] \nabla^2 \int_V d^3\boldsymbol{r}' \, \boldsymbol{F}(\boldsymbol{r}') G(\boldsymbol{r}, \boldsymbol{r}') \\
&= -\nabla^2 \boldsymbol{H}(\boldsymbol{r})
\end{aligned} \tag{19.2.85}$$

where

$$\boldsymbol{H}(\boldsymbol{r}) = [1/(4\pi)] \int_V d^3\boldsymbol{r}' \, \boldsymbol{F}(\boldsymbol{r}') G(\boldsymbol{r}, \boldsymbol{r}'). \tag{19.2.86}$$

Invoke again the vector identity

$$-\nabla^2 \boldsymbol{H}(\boldsymbol{r}) = \nabla \times [\nabla \times \boldsymbol{H}(\boldsymbol{r})] - \nabla[\nabla \cdot \boldsymbol{H}(\boldsymbol{r})]. \tag{19.2.87}$$

It follows that

$$\boldsymbol{F}(\boldsymbol{r}) = \nabla \times [\nabla \times \boldsymbol{H}(\boldsymbol{r})] - \nabla[\nabla \cdot \boldsymbol{H}(\boldsymbol{r})], \tag{19.2.88}$$

and therefore (2.74) holds with the definitions

$$\phi(\boldsymbol{r}) = \nabla \cdot \boldsymbol{H}(\boldsymbol{r}), \tag{19.2.89}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \nabla \times \boldsymbol{H}(\boldsymbol{r}). \tag{19.2.90}$$

It remains to work out computationally useful expressions for $\phi(\boldsymbol{r})$ and $\boldsymbol{A}(\boldsymbol{r})$. Doing so requires a flurry of vector manipulations. Begin with $\phi(\boldsymbol{r})$. According to (2.86) and (2.89) it can be written as

$$\phi(\boldsymbol{r}) = [1/(4\pi)] \int_V d^3\boldsymbol{r}' \; \nabla \cdot [\boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')]. \tag{19.2.91}$$

Manipulate the integrand in (2.91) to find the result

$$\begin{aligned} \nabla \cdot [\boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')] &= \boldsymbol{F}(\boldsymbol{r}') \cdot \nabla G(\boldsymbol{r},\boldsymbol{r}')] = -\boldsymbol{F}(\boldsymbol{r}') \cdot \nabla' G(\boldsymbol{r},\boldsymbol{r}')] \\ &= -\nabla' \cdot [\boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')] + G(\boldsymbol{r},\boldsymbol{r}')\nabla' \cdot \boldsymbol{F}(\boldsymbol{r}'). \end{aligned} \tag{19.2.92}$$

Employ this result in (2.91) to rewrite it in the form

$$\phi(\boldsymbol{r}) = [1/(4\pi)] \int_V d^3\boldsymbol{r}' \; \{-\nabla' \cdot [\boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')] + G(\boldsymbol{r},\boldsymbol{r}')\nabla' \cdot \boldsymbol{F}(\boldsymbol{r}')\}. \tag{19.2.93}$$

Finally, use the divergence theorem to transform the first term on the right side of (2.91) to yield the result

$$\phi(\boldsymbol{r}) = -[1/(4\pi)] \int_S dS' \; \boldsymbol{n}' \cdot \boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}') + [1/(4\pi)] \int_V d^3\boldsymbol{r}' \; G(\boldsymbol{r},\boldsymbol{r}')\nabla' \cdot \boldsymbol{F}(\boldsymbol{r}'), \tag{19.2.94}$$

in agreement with (2.77).

The case of $\boldsymbol{A}(\boldsymbol{r})$ requires somewhat more effort. Combining (2.86) and (2.90) gives the result

$$\boldsymbol{A}(\boldsymbol{r}) = [1/(4\pi)] \int_V d^3\boldsymbol{r}' \; \nabla \times [\boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')]. \tag{19.2.95}$$

Manipulate the integrand in (2.95) to find the result

$$\begin{aligned} \nabla \times [\boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')] &= [\nabla G(\boldsymbol{r},\boldsymbol{r}')] \times \boldsymbol{F}(\boldsymbol{r}') = -\boldsymbol{F}(\boldsymbol{r}') \times \nabla G(\boldsymbol{r},\boldsymbol{r}') \\ &= \boldsymbol{F}(\boldsymbol{r}') \times \nabla' G(\boldsymbol{r},\boldsymbol{r}'). \end{aligned} \tag{19.2.96}$$

There is also the vector identity

$$\nabla' \times [\boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')] = G(\boldsymbol{r},\boldsymbol{r}')\nabla' \times \boldsymbol{F}(\boldsymbol{r}') - \boldsymbol{F}(\boldsymbol{r}') \times \nabla' G(\boldsymbol{r},\boldsymbol{r}'). \tag{19.2.97}$$

Combining (2.96) and (2.97) gives the result

$$\nabla \times [\boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')] = G(\boldsymbol{r},\boldsymbol{r}')\nabla' \times \boldsymbol{F}(\boldsymbol{r}') - \nabla' \times [\boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')]. \tag{19.2.98}$$

Employ this result in (2.95) to rewrite it in the form

$$\boldsymbol{A}(\boldsymbol{r}) = [1/(4\pi)] \int_V d^3\boldsymbol{r}' \; G(\boldsymbol{r},\boldsymbol{r}')\nabla' \times \boldsymbol{F}(\boldsymbol{r}') - [1/(4\pi)] \int_V d^3\boldsymbol{r}' \; \nabla' \times [\boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')]. \tag{19.2.99}$$

Now work on the second integral appearing on the right side of (2.99). Let $\boldsymbol{c}$ be any constant vector. By the divergence theorem there is the relation

$$\int_V d^3\boldsymbol{r}' \; \nabla' \cdot [\boldsymbol{c} \times G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')] = \int_S dS' \; \boldsymbol{n}' \cdot [\boldsymbol{c} \times G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')]. \tag{19.2.100}$$

There is also the vector identity

$$\begin{aligned}\boldsymbol{n}'\cdot[\boldsymbol{c}\times G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')] &= -\boldsymbol{n}'\cdot[G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')\times\boldsymbol{c}]\\ &= -[\boldsymbol{n}'\times G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')]\cdot\boldsymbol{c}\\ &= -\boldsymbol{c}\cdot[\boldsymbol{n}'\times G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')]. \end{aligned}\tag{19.2.101}$$

Consequently, (2.100) can be rewritten in the form

$$\int_V d^3\boldsymbol{r}'\ \nabla'\cdot[\boldsymbol{c}\times G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')] = -\boldsymbol{c}\cdot\int_S dS'\ [\boldsymbol{n}'\times G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')].\tag{19.2.102}$$

Next manipulate the integrand on the left side of (2.102) to find the result

$$\begin{aligned}\nabla'\cdot[\boldsymbol{c}\times G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')] &= -\nabla'\cdot[G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')\times\boldsymbol{c}]\\ &= -\{\nabla'\times[G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')]\}\cdot\boldsymbol{c}\\ &= -\boldsymbol{c}\cdot\{\nabla'\times[G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')]\}. \end{aligned}\tag{19.2.103}$$

Therefore (2.102) can be rewritten as

$$-\boldsymbol{c}\cdot\int_V d^3\boldsymbol{r}'\ \nabla'\times[G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')] = -\boldsymbol{c}\cdot\int_S dS'\ [\boldsymbol{n}'\times G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')],\tag{19.2.104}$$

from which it follows, because $\boldsymbol{c}$ is arbitrary, that

$$\int_V d^3\boldsymbol{r}'\ \nabla'\times[G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')] = \int_S dS'\ [\boldsymbol{n}'\times G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')].\tag{19.2.105}$$

The last step is to employ (2.105) in (2.99) to obtain the final result

$$\boldsymbol{A}(\boldsymbol{r}) = [1/(4\pi)]\int_V d^3\boldsymbol{r}'\ G(\boldsymbol{r},\boldsymbol{r}')\nabla'\times\boldsymbol{F}(\boldsymbol{r}') - [1/(4\pi)]\int_S dS'\ [\boldsymbol{n}'\times G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')],\tag{19.2.106}$$

in agreement with (2.78).

It still remains to be shown that, for the definitions made, $\nabla\cdot\boldsymbol{A}(\boldsymbol{r}) = 0$. Look at (2.90). Since the divergence of a curl vanishes, when suitable smoothness conditions are met by the functions involved, it follows that under these conditions $\boldsymbol{A}(\boldsymbol{r})$ as given by (2.90), and therefore also by (2.106), is indeed divergence free. From (2.86) we see that the analytic properties of $\boldsymbol{H}(\boldsymbol{r})$ are determined by those of $\boldsymbol{F}(\boldsymbol{r})$. In general $\boldsymbol{H}(\boldsymbol{r})$ will be smoother than $\boldsymbol{F}(\boldsymbol{r})$. See Appendix F. Therefore, under mild conditions on $\boldsymbol{F}(\boldsymbol{r})$, the vector potential $\boldsymbol{A}(\boldsymbol{r})$ will be divergence free.

Exercises

19.2.1. Verify the expansions (2.11) and (2.17).

19.2.2. Verify (2.20).

19.2.3. Verify the identity (2.23) and its use to evaluate the integral (2.24).

19.2.4. Verify that $\boldsymbol{A}(\boldsymbol{r})$ as given by (2.33) satisfies (2.31).

19.2.5. The purpose of this exercise is to verify (2.37) using (2.27).

19.2.6. The purpose of this exercise is to verify (2.38) using the definition (2.39).

19.2.7. The purpose of this exercise is to verify (2.42) using the definition (2.41).

19.2.8. Verify (2.43).

19.2.9. Verify (2.44).

19.2.10. Show that the integral (2.47) can be written in the form

$$[g/(4\pi)]\int_{\boldsymbol{r}_A}^{\boldsymbol{r}_B} d\boldsymbol{r}_d \times (\boldsymbol{r}-\boldsymbol{r}_d)/|\boldsymbol{r}-\boldsymbol{r}_d|^3 = -[g/(4\pi)]\int_{\boldsymbol{r}_A}^{\boldsymbol{r}_B} d\boldsymbol{r}_d \times \nabla[1/|\boldsymbol{r}-\boldsymbol{r}_d|]. \quad (19.2.107)$$

19.2.11. Verify (2.50).

19.2.12. Nature of thin solenoid and nature of field at the end of a thin solenoid.

19.2.13. The purpose of this exercise is to verify (2.54).

19.2.14. The purpose of this exercise is to verify (2.62).

19.2.15. Evaluate $\boldsymbol{A}_s(\boldsymbol{r};\boldsymbol{r}_A,\boldsymbol{m})$ as given by (2.63) for the case

$$\boldsymbol{r}_A = 0 \quad (19.2.108)$$

and

$$\boldsymbol{m} = \boldsymbol{e}_z. \quad (19.2.109)$$

Show, using spherical coordinates, that in this case $\boldsymbol{A}_s(\boldsymbol{r};\boldsymbol{r}_A,\boldsymbol{m})$ has only a ϕ component A^s_ϕ given by

$$A^s_\phi = [g/(4\pi)](1+\cos\theta)/[r\sin\theta]. \quad (19.2.110)$$

Verify that A^s_ϕ is singular on the positive z axis, but not on the negative z axis. Show, by explicit calculation, that

$$\nabla \times \boldsymbol{A}_s(\boldsymbol{r};0,\boldsymbol{e}_z) = -[g/(4\pi)][\boldsymbol{r}/|\boldsymbol{r}|^3], \quad (19.2.111)$$

as expected.

19.2.16. Exercise on the singularity structure of the vector potential for a straight half-infinite Dirac string.

19.2.17. Let $\boldsymbol{A}_s(\boldsymbol{r};\boldsymbol{r}_A,\boldsymbol{m})$ and $\boldsymbol{A}_s(\boldsymbol{r};\boldsymbol{r}_A,\boldsymbol{m}')$ be equal strength monopole vector potentials produced by straight-line strings both originating at $\boldsymbol{r}_A$ but extending to infinity in the directions $\boldsymbol{m}$ and $\boldsymbol{m}'$. See (2.63). Show that both produce the same magnetic field (2.55) at points off the strings. Show that these vector potentials are related by a gauge transformation.

19.2.18. Verify (2.72) and (2.73).

19.2.19. Suppose the vector field $\boldsymbol{F}(\boldsymbol{r})$ is specified in some volume V. Surround this volume by a thin shell Σ. Extend $\boldsymbol{F}(\boldsymbol{r})$ to all of space by requiring that it vanish outside Σ and go to zero smoothly within Σ. It is a standard result in analysis that this can be done in such a way that $\boldsymbol{F}(\boldsymbol{r})$ will have as many derivatives as desired in Σ. Find formulas for $\phi(\boldsymbol{r})$ and $\boldsymbol{A}(\boldsymbol{r})$ in this case. Now let the shell shrink to zero thickness while keeping V unchanged so that Σ becomes the surface S. Show that the relations (2.74), (2.77), and (2.78) continue to give $\boldsymbol{F}(\boldsymbol{r})$ for $\boldsymbol{r} \in V$, and give $\boldsymbol{F}(\boldsymbol{r}) = 0$ for $\boldsymbol{r} \notin V$.

19.2.20. Suppose that $\boldsymbol{F}(\boldsymbol{r})$ is globally defined and falls off at infinity as fast as $1/|\boldsymbol{r}|^2$. Show that, when the surface S in (2.77) and (2.78) is taken to infinity, the surface integrals then vanish. Consequently, (2.77) and (2.78) then take the form

$$\phi(\boldsymbol{r}) = [1/(4\pi)] \int d^3\boldsymbol{r}'\, G(\boldsymbol{r}, \boldsymbol{r}')\nabla' \cdot \boldsymbol{F}(\boldsymbol{r}'), \tag{19.2.112}$$

$$\boldsymbol{A}(\boldsymbol{r}) = [1/(4\pi)] \int d^3\boldsymbol{r}'\, G(\boldsymbol{r}, \boldsymbol{r}')\nabla' \times \boldsymbol{F}(\boldsymbol{r}'). \tag{19.2.113}$$

Suppose that $\boldsymbol{F}(\boldsymbol{r})$ has the Fourier representation

$$\boldsymbol{F}(\boldsymbol{r}) = \int d^3\boldsymbol{k}\, \exp(i\boldsymbol{k} \cdot \boldsymbol{r})\, \tilde{\boldsymbol{F}}(\boldsymbol{k}). \tag{19.2.114}$$

Such a representation is possible in any number of dimensions, and its existence is a consequence of the completeness of the unitary representations of the translation part of the Euclidean group. Show that there are the relations

$$\boldsymbol{\nabla} \cdot \boldsymbol{F}(\boldsymbol{r}) = i \int d^3\boldsymbol{k}\, \exp(i\boldsymbol{k} \cdot \boldsymbol{r})\, \boldsymbol{k} \cdot \tilde{\boldsymbol{F}}(\boldsymbol{k}), \tag{19.2.115}$$

$$\boldsymbol{\nabla} \times \boldsymbol{F}(\boldsymbol{r}) = i \int d^3\boldsymbol{k}\, \exp(i\boldsymbol{k} \cdot \boldsymbol{r})\, \boldsymbol{k} \times \tilde{\boldsymbol{F}}(\boldsymbol{k}). \tag{19.2.116}$$

Consequently, if the functions $\boldsymbol{\nabla} \cdot \boldsymbol{F}(\boldsymbol{r})$ and $\boldsymbol{\nabla} \times \boldsymbol{F}(\boldsymbol{r})$ are assumed known, then, by the Fourier inversion theorem, the functions $\boldsymbol{k} \cdot \tilde{\boldsymbol{F}}(\boldsymbol{k})$ and $\boldsymbol{k} \times \tilde{\boldsymbol{F}}(\boldsymbol{k})$ are also known. Recall the vector identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\, (\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\, (\boldsymbol{a} \cdot \boldsymbol{b}). \tag{19.2.117}$$

Use this identity to show that

$$\boldsymbol{k} \times (\boldsymbol{k} \times \tilde{\boldsymbol{F}}) = \boldsymbol{k}\, (\boldsymbol{k} \cdot \tilde{\boldsymbol{F}}) - \tilde{\boldsymbol{F}}\, (\boldsymbol{k} \cdot \boldsymbol{k}), \tag{19.2.118}$$

and therefore

$$\tilde{\boldsymbol{F}} = [1/(\boldsymbol{k} \cdot \boldsymbol{k})][\boldsymbol{k}(\boldsymbol{k} \cdot \tilde{\boldsymbol{F}})] - [1/(\boldsymbol{k} \cdot \boldsymbol{k})][\boldsymbol{k} \times (\boldsymbol{k} \times \tilde{\boldsymbol{F}})]. \tag{19.2.119}$$

Thus, the function $\tilde{\boldsymbol{F}}(\boldsymbol{k})$ is known if the functions $\boldsymbol{k} \cdot \tilde{\boldsymbol{F}}(\boldsymbol{k})$ and $\boldsymbol{k} \times \tilde{\boldsymbol{F}}(\boldsymbol{k})$ are known. Correspondingly, the function $\boldsymbol{F}(\boldsymbol{r})$ is determined if the functions $\boldsymbol{\nabla} \cdot \boldsymbol{F}(\boldsymbol{r})$ and $\boldsymbol{\nabla} \times \boldsymbol{F}(\boldsymbol{r})$

are assumed known. Finally, we note that the identity (2.117) may be viewed as a Lie algebraic relation for the cross-product Lie algebra. See Section 3.7.4. From Exercise 3.7.30 we know that the cross-product Lie algebra is equivalent to $so(3)$, and therefore (2.117) is also a property of $so(3)$. Finally, $so(3)$ is a subalgebra of the Lie algebra of the three-dimensional Euclidean group. Thus, the fact that a vector field in three dimensions is specified, if its divergence and curl are known, is a consequence of the properties of the the three-dimensional Euclidean group.

19.3 Construction of Kernels $\boldsymbol{G}^n$ and $\boldsymbol{G}^t$

Let us apply the results of the previous section to the case of a magnetic field $\boldsymbol{B}(\boldsymbol{r})$ in a volume V under the assumption that there are no sources in V. See (1.1) and (1.2). As stated earlier, this would be the case of interest for charged particles propagating through an evacuated beam pipe. In this circumstance we may use (2.74), (2.81), and (2.82) to write

$$\boldsymbol{B}(\boldsymbol{r}) = -\nabla\phi^n(\boldsymbol{r}) + \nabla \times \boldsymbol{A}^t(\boldsymbol{r}) \text{ for } \boldsymbol{r} \in V \tag{19.3.1}$$

with

$$\phi^n(\boldsymbol{r}) = -[1/(4\pi)] \int_S dS' \, \boldsymbol{n}' \cdot \boldsymbol{B}(\boldsymbol{r}')G(\boldsymbol{r}, \boldsymbol{r}'), \tag{19.3.2}$$

$$\boldsymbol{A}^t(\boldsymbol{r}) = -[1/(4\pi)] \int_S dS' \, [\boldsymbol{n}' \times \boldsymbol{B}(\boldsymbol{r}')]G(\boldsymbol{r}, \boldsymbol{r}'). \tag{19.3.3}$$

Here the superscripts n and t denote *normal* and *tangential* since the quantities so denoted involve normal and tangential components of $\boldsymbol{B}$.

The relations (3.1) through (3.3) could be employed if one wished to integrate Newton's equations of motion, and also find Taylor maps based on these equations, for all that would then be required is the magnetic field $\boldsymbol{B}(\boldsymbol{r})$. See, for example, the equations of motion (1.6.68) and (1.6.69). However, if one wishes instead to employ a Hamiltonian formulation in order to reap the benefits of a symplectic formulation, then it is necessary to have the magnetic field specified *entirely* in terms of a vector potential rather than in terms of both a scalar and vector potential as in (3.1). What we need is a vector potential $\boldsymbol{A}^n(\boldsymbol{r})$ such that

$$\nabla \times \boldsymbol{A}^n(\boldsymbol{r}) = -\nabla\phi^n(\boldsymbol{r}). \tag{19.3.4}$$

Then, with the definition

$$\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{A}^n(\boldsymbol{r}) + \boldsymbol{A}^t(\boldsymbol{r}), \tag{19.3.5}$$

there would be the result

$$\boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}). \tag{19.3.6}$$

The construction of an $\boldsymbol{A}^n(\boldsymbol{r})$ that satisfies (3.4) can be accomplished with the aid of the Dirac monopole vector potential. Inspection of $\phi^n(\boldsymbol{r})$, as given by (3.2), shows that it appears to arise from a distribution of magnetic monopoles described by a magnetic charge surface density spread over the surface S. Therefore, it should be possible to find an equivalent vector potential based on the vector potential for a magnetic monopole.

Let us make this idea precise. Define $\boldsymbol{B}^n$ by the rule

$$\boldsymbol{B}^n = -\nabla\phi^n \tag{19.3.7}$$

so that the $\boldsymbol{A}^n$ that we seek satisfies

$$\nabla \times \boldsymbol{A}^n = \boldsymbol{B}^n. \tag{19.3.8}$$

Combining (3.2) and (3.7) gives the result

$$\boldsymbol{B}^n = [1/(4\pi)] \int_S dS' \, \boldsymbol{n}' \cdot \boldsymbol{B}(\boldsymbol{r}')\nabla G(\boldsymbol{r}, \boldsymbol{r}'). \tag{19.3.9}$$

From (2.83) we know that

$$\nabla G(\boldsymbol{r}, \boldsymbol{r}') = -(\boldsymbol{r} - \boldsymbol{r}')/|\boldsymbol{r} - \boldsymbol{r}'|^3. \tag{19.3.10}$$

But, from (2.72), we also have the relation

$$(4\pi/g)\nabla \times \boldsymbol{A}_s(\boldsymbol{r}; \boldsymbol{r}', \boldsymbol{m}') = -[(\boldsymbol{r} - \boldsymbol{r}')/|\boldsymbol{r} - \boldsymbol{r}'|^3]. \tag{19.3.11}$$

Define a quantity $\boldsymbol{K}(\boldsymbol{r}; \boldsymbol{r}', \boldsymbol{m}')$ by the rule

$$\begin{aligned} \boldsymbol{K}(\boldsymbol{r}; \boldsymbol{r}', \boldsymbol{m}') &= (4\pi/g)\boldsymbol{A}_s(\boldsymbol{r}; \boldsymbol{r}', \boldsymbol{m}') \\ &= [\boldsymbol{m}' \times (\boldsymbol{r} - \boldsymbol{r}')]/\{|\boldsymbol{r} - \boldsymbol{r}'|[|\boldsymbol{r} - \boldsymbol{r}'| - \boldsymbol{m}' \cdot (\boldsymbol{r} - \boldsymbol{r}')]\}. \end{aligned} \tag{19.3.12}$$

See (2.63). In view of (3.10) through (3.12), we have established the key relation

$$\nabla G(\boldsymbol{r}, \boldsymbol{r}') = \nabla \times \boldsymbol{K}(\boldsymbol{r}; \boldsymbol{r}', \boldsymbol{m}'). \tag{19.3.13}$$

See exercise 2.15 for a specific instance of this relation.

We are almost done. Insertion of (3.13) into (3.9) gives the result

$$\begin{aligned} \boldsymbol{B}^n &= [1/(4\pi)] \int_S dS' \, \boldsymbol{n}' \cdot \boldsymbol{B}(\boldsymbol{r}')\nabla \times \boldsymbol{K}(\boldsymbol{r}; \boldsymbol{r}', \boldsymbol{m}') \\ &= [1/(4\pi)]\nabla \times \int_S dS' \, \boldsymbol{n}' \cdot \boldsymbol{B}(\boldsymbol{r}')\boldsymbol{K}(\boldsymbol{r}; \boldsymbol{r}', \boldsymbol{m}'). \end{aligned} \tag{19.3.14}$$

Comparison of (3.8) and (3.14) shows that we may make the definition

$$\boldsymbol{A}^n(\boldsymbol{r}) = [1/(4\pi)] \int_S dS' \, \boldsymbol{n}' \cdot \boldsymbol{B}(\boldsymbol{r}')\boldsymbol{K}(\boldsymbol{r}; \boldsymbol{r}', \boldsymbol{m}'). \tag{19.3.15}$$

In evaluating the integral (3.15) it necessary to specify $\boldsymbol{m}'(\boldsymbol{r}')$, the direction of the straight half-infinite Dirac string, as $\boldsymbol{r}'$ varies over S. There is considerable freedom in doing so, and different choices simply result in different gauges for $\boldsymbol{A}^n(\boldsymbol{r})$. The major consideration is that no string intersect the volume V because it is desirable that $\boldsymbol{A}^n(\boldsymbol{r})$ be analytic for $\boldsymbol{r} \in V$. For many geometries a convenient choice is to require that $\boldsymbol{m}'(\boldsymbol{r}')$ be normal to and point outward from S,

$$\boldsymbol{m}'(\boldsymbol{r}') = \boldsymbol{n}'(\boldsymbol{r}'). \tag{19.3.16}$$

With this choice we may write

$$\boldsymbol{A}^n(\boldsymbol{r}) = \int_S dS' \; B_n(\boldsymbol{r}')\boldsymbol{G}^n(\boldsymbol{r},\boldsymbol{r}') \tag{19.3.17}$$

where

$$B_n(\boldsymbol{r}') = \boldsymbol{n}' \cdot \boldsymbol{B}(\boldsymbol{r}'), \tag{19.3.18}$$

and $\boldsymbol{G}^n(\boldsymbol{r},\boldsymbol{r}')$ is the kernel

$$\begin{aligned} \boldsymbol{G}^n(\boldsymbol{r},\boldsymbol{r}') &= [1/(4\pi)]\boldsymbol{K}(\boldsymbol{r};\boldsymbol{r}',\boldsymbol{n}') \\ &= \{\boldsymbol{n}'(\boldsymbol{r}') \times (\boldsymbol{r}-\boldsymbol{r}')\}/\{4\pi|\boldsymbol{r}-\boldsymbol{r}'|[|\boldsymbol{r}-\boldsymbol{r}'| - \boldsymbol{n}'(\boldsymbol{r}')\cdot(\boldsymbol{r}-\boldsymbol{r}')]\}. \end{aligned} \tag{19.3.19}$$

We have derived the relations (1.7) and (1.10).

At this point we can take pleasure in observing that $\boldsymbol{A}^n(\boldsymbol{r})$ and $\boldsymbol{G}^n(\boldsymbol{r},\boldsymbol{r}')$, as given by (3.17) and (3.19) or (1.7) and (1.10), have several desirable properties: First, as long as the Dirac strings for $\boldsymbol{r}' \in S$ do not intersect V, the functions $\boldsymbol{G}^n(\boldsymbol{r},\boldsymbol{r}')$, for every $\boldsymbol{r}' \in S$, are analytic in $\boldsymbol{r}$ for all $\boldsymbol{r} \in V$. It follows from (3.17), under mild conditions on $B_n(\boldsymbol{r}')$ for $\boldsymbol{r}' \in S$, that $\boldsymbol{A}^n(\boldsymbol{r})$ is analytic in V. Second, since the kernel $\boldsymbol{G}^n(\boldsymbol{r},\boldsymbol{r}')$ is essentially the vector potential for a Dirac magnetic monopole, see (3.12) and (3.19), it has, for $\boldsymbol{r} \in V$, the properties

$$\nabla \cdot [\boldsymbol{G}^n(\boldsymbol{r},\boldsymbol{r}')] = 0, \tag{19.3.20}$$

$$\nabla \times [\nabla \times \boldsymbol{G}^n(\boldsymbol{r},\boldsymbol{r}')] = 0. \tag{19.3.21}$$

See (2.71) and (2.73). It follows from (3.17), again under mild conditions on $B_n(\boldsymbol{r}')$, that $\boldsymbol{A}^n(\boldsymbol{r})$ has these same properties,

$$\nabla \cdot [\boldsymbol{A}^n(\boldsymbol{r})] = 0, \tag{19.3.22}$$

$$\nabla \times [\nabla \times \boldsymbol{A}^n(\boldsymbol{r})] = 0. \tag{19.3.23}$$

In practical applications, the surface values $B_n(\boldsymbol{r}')$ will only be known approximately, and the integrals (3.17) may be evaluated numerically with limited precision. It is comforting to know that, nevertheless, the resulting $\boldsymbol{A}^n(\boldsymbol{r})$ will be analytic in V and will satisfy the relations (3.22) and (3.23) exactly no matter what errors are present in the surface values $B_n(\boldsymbol{r}')$ and no matter how poorly the integrals (3.17) are evaluated. All that matters is that the kernel $\boldsymbol{G}^n$ be evaluated to high precision.

What can be said about the properties of $\boldsymbol{A}^t(\boldsymbol{r})$ as given by (3.3)? Just as is the case for $\boldsymbol{A}^n(\boldsymbol{r})$, we would like $\boldsymbol{A}^t(\boldsymbol{r})$ to be analytic in V and to satisfy properties analogous to (3.22) and (3.23) no matter how poorly the integrals (3.3) are evaluated. As the expression (3.3) for $\boldsymbol{A}^t(\boldsymbol{r})$ stands, this is not the case. However, we can transform (3.3) into a form that meets all our hopes.

Since, by assumption, $\boldsymbol{B}(\boldsymbol{r}')$ is curl free for $\boldsymbol{r}' \in V$, there exists a scalar potential $\psi(\boldsymbol{r}')$ such that

$$\boldsymbol{B}(\boldsymbol{r}') = \nabla'\psi(\boldsymbol{r}'). \tag{19.3.24}$$

[Note, by convention, we have used a minus sign in (2.3) and a plus sign in (3.24). See also (13.2.1).] Consequently, (3.3) can be rewritten in the form

$$\boldsymbol{A}^t(\boldsymbol{r}) = -[1/(4\pi)] \int_S dS' \, [\boldsymbol{n}' \times \nabla' \psi(\boldsymbol{r}')] G(\boldsymbol{r}, \boldsymbol{r}'). \tag{19.3.25}$$

[Note that a knowledge of the tangential component of $\nabla'\psi(\boldsymbol{r}')$, which is what is involved in (3.25) and is equivalent to a knowledge of $\psi(\boldsymbol{r}')$ on S, is in turn equivalent to a knowledge of the tangential component of $\boldsymbol{B}(\boldsymbol{r}')$ on S under the assumption that $\boldsymbol{B}(\boldsymbol{r}')$ is curl free.] Next observe that there is the identity

$$[\nabla'\psi(\boldsymbol{r}')]G(\boldsymbol{r}, \boldsymbol{r}') = \nabla'[\psi(\boldsymbol{r}')G(\boldsymbol{r}, \boldsymbol{r}')] - \psi(\boldsymbol{r}')\nabla' G(\boldsymbol{r}, \boldsymbol{r}'). \tag{19.3.26}$$

Therefore (3.25) can also be written in the form

$$\begin{aligned} \boldsymbol{A}^t(\boldsymbol{r}) &= -[1/(4\pi)] \int_S dS' \, \{\boldsymbol{n}' \times \nabla'[\psi(\boldsymbol{r}')G(\boldsymbol{r}, \boldsymbol{r}')]\} \\ &\quad +[1/(4\pi)] \int_S dS' \, \{\boldsymbol{n}' \times [\psi(\boldsymbol{r}')\nabla' G(\boldsymbol{r}, \boldsymbol{r}')]\}. \end{aligned} \tag{19.3.27}$$

It can be shown that the first integral on the right side of (3.27) vanishes,

$$-[1/(4\pi)] \int_S dS' \, \{\boldsymbol{n}' \times \nabla'[\psi(\boldsymbol{r}')G(\boldsymbol{r}, \boldsymbol{r}')]\} = 0. \tag{19.3.28}$$

See Exercise 3.1. Moreover, the second integral can be rewritten in the form

$$[1/(4\pi)] \int_S dS' \, \{\boldsymbol{n}' \times [\psi(\boldsymbol{r}')\nabla' G(\boldsymbol{r}, \boldsymbol{r}')]\} = [1/(4\pi)] \int_S dS' \, \psi(\boldsymbol{r}')[\boldsymbol{n}' \times \nabla' G(\boldsymbol{r}, \boldsymbol{r}')]. \tag{19.3.29}$$

Consequently $\boldsymbol{A}^t(\boldsymbol{r})$ can also be written in the form

$$\boldsymbol{A}^t(\boldsymbol{r}) = [1/(4\pi)] \int_S dS' \, \psi(\boldsymbol{r}')[\boldsymbol{n}' \times \nabla' G(\boldsymbol{r}, \boldsymbol{r}')]. \tag{19.3.30}$$

Finally, let $\boldsymbol{G}^t(\boldsymbol{r}, \boldsymbol{r}')$ be the kernel

$$\boldsymbol{G}^t(\boldsymbol{r}, \boldsymbol{r}') = [1/(4\pi)][\boldsymbol{n}'(\boldsymbol{r}') \times \nabla' G(\boldsymbol{r}, \boldsymbol{r}')]. \tag{19.3.31}$$

With this definition, $\boldsymbol{A}^t(\boldsymbol{r})$ takes the final form

$$\boldsymbol{A}^t(\boldsymbol{r}) = \int_S dS' \, \psi(\boldsymbol{r}')\boldsymbol{G}^t(\boldsymbol{r}, \boldsymbol{r}'). \tag{19.3.32}$$

And working out (3.31) explicitly gives the result

$$\boldsymbol{G}^t(\boldsymbol{r}, \boldsymbol{r}') = [\boldsymbol{n}'(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')]/[4\pi|\boldsymbol{r} - \boldsymbol{r}'|^3]. \tag{19.3.33}$$

We have derived the relations (1.8) and (1.11).

At this point we should verify that we have achieved our desired goals. First, it is evident from (3.33) that $\boldsymbol{G}^t(\boldsymbol{r},\boldsymbol{r}')$ is analytic in the components of $\boldsymbol{r}$ for $\boldsymbol{r} \in V$ and $\boldsymbol{r}' \in S$. Therefore, from the representation (3.32), we see that, under mild conditions on $\psi(\boldsymbol{r}')$, $\boldsymbol{A}^t(\boldsymbol{r})$ will be analytic in V. Next, let us compute $\nabla \cdot \boldsymbol{G}^t(\boldsymbol{r},\boldsymbol{r}')$. Recall the vector identity

$$\nabla \cdot (\boldsymbol{C} \times \boldsymbol{D}) = \boldsymbol{D} \cdot (\nabla \times \boldsymbol{C}) - \boldsymbol{C} \cdot (\nabla \times \boldsymbol{D}). \tag{19.3.34}$$

From this identity, and using (3.31), it follows that

$$\begin{aligned}\nabla \cdot \boldsymbol{G}^t(\boldsymbol{r},\boldsymbol{r}') &= -[1/(4\pi)]\boldsymbol{n}'(\boldsymbol{r}') \cdot \{\nabla \times [\nabla' G(\boldsymbol{r},\boldsymbol{r}')]\} \\ &= [1/(4\pi)]\boldsymbol{n}'(\boldsymbol{r}') \cdot \{\nabla \times [\nabla G(\boldsymbol{r},\boldsymbol{r}')]\} = 0.\end{aligned} \tag{19.3.35}$$

Also, it is evident from (2.84) and (3.31) that

$$\nabla^2 \boldsymbol{G}^t(\boldsymbol{r},\boldsymbol{r}') = 0 \text{ for } \boldsymbol{r} \text{ within } V \text{ and } \boldsymbol{r}' \in S. \tag{19.3.36}$$

Again invoke the vector identity

$$\nabla \times (\nabla \times \boldsymbol{C}) = \nabla(\nabla \cdot \boldsymbol{C}) - \nabla^2 \boldsymbol{C}. \tag{19.3.37}$$

When applied to $\boldsymbol{G}^t(\boldsymbol{r},\boldsymbol{r}')$, in view of (3.35) and (3.36), it yields the relation

$$\nabla \times [\nabla \times \boldsymbol{G}^t(\boldsymbol{r},\boldsymbol{r}')] = 0 \text{ for } \boldsymbol{r} \text{ within } V \text{ and } \boldsymbol{r}' \in S. \tag{19.3.38}$$

We have seen that the kernel $\boldsymbol{G}^t(\boldsymbol{r},\boldsymbol{r}')$ satisfies the relations (3.35) and (3.38), and note that these relations are analogous to the relations (3.20) and (3.21) for $\boldsymbol{G}^n(\boldsymbol{r},\boldsymbol{r}')$. It follows, by the same reasoning used in the case of $\boldsymbol{G}^n(\boldsymbol{r},\boldsymbol{r}')$ and $\boldsymbol{A}^n(\boldsymbol{r})$, that $\boldsymbol{A}^t(\boldsymbol{r})$ satisfies the relations

$$\nabla \cdot [\boldsymbol{A}^t(\boldsymbol{r})] = 0, \tag{19.3.39}$$

$$\nabla \times [\nabla \times \boldsymbol{A}^t(\boldsymbol{r})] = 0, \tag{19.3.40}$$

and these relations hold exactly even in the presence of errors in the surface values $\psi(\boldsymbol{r}')$ and no matter how poorly the integrals (3.32) are evaluated. As before, all that matters is that the kernel $\boldsymbol{G}^t$ be evaluated to high precision.

The last step is to utilize (3.5) and (3.6). Since $\boldsymbol{A}^n(\boldsymbol{r})$ and $\boldsymbol{A}^t(\boldsymbol{r})$ are both analytic in V, $\boldsymbol{A}(\boldsymbol{r})$ will be analytic in V. And since (3.22), (3.23), (3.39), and (3.40) hold, the same will be true of $\boldsymbol{A}(\boldsymbol{r})$,

$$\nabla \cdot [\boldsymbol{A}(\boldsymbol{r})] = 0, \tag{19.3.41}$$

$$\nabla \times [\nabla \times \boldsymbol{A}(\boldsymbol{r})] = 0, \tag{19.3.42}$$

and these relations will again hold exactly even in the presence of errors in the surface values and no matter how poorly the relevant integrals are evaluated. Finally, in view of (3.6), the Maxwell equation

$$\nabla \cdot \boldsymbol{B} = 0 \tag{19.3.43}$$

will be satisfied exactly. And, in view of (3.6) and (3.42), the second Maxwell equation

$$\nabla \times \boldsymbol{B} = 0 \tag{19.3.44}$$

will also be satisfied exactly.

Exercises

19.3.1. The purpose of this exercise is to verify the relation (3.28).

19.3.2. At the beginning of this section it was mentioned that (3.1) through (3.3) could be used to integrate Newton's equations of motion in terms of $\boldsymbol{B}(\boldsymbol{r})$. However the $\boldsymbol{B}(\boldsymbol{r})$ obtained using (3.1) is not guaranteed to satisfy the Maxwell equations if there are errors in surface values and/or the integrals are not evaluated accurately. Verify that, in this regard, there is no difficulty in the use of (3.2) by showing that it is guaranteed to satisfy

$$\nabla^2 \phi^n(\boldsymbol{r}) = 0, \tag{19.3.45}$$

and therefore (3.43) is satisfied. Show that if (3.3) is replaced by (3.32), then (3.44) is also guaranteed.

19.3.3. Suppose $\boldsymbol{B}(\boldsymbol{r})$ is source free in a volume V bounded by a surface S, as in (1.1) and (1.2), and suppose $B_n(\boldsymbol{r}')$ and $\psi(\boldsymbol{r}')$ are known on S. The aim of this exercise is to compute $\boldsymbol{B}(\boldsymbol{r})$ in terms of $B_n(\boldsymbol{r}')$ and $\psi(\boldsymbol{r}')$ using the representation given by (1.3), (1.6) through (1.8), (1.10), and (1.11). Verify that

$$\nabla \times \boldsymbol{G}^n(\mathbf{r}, \mathbf{r}') = [1/(4\pi)]\nabla G(\mathbf{r}, \mathbf{r}') \tag{19.3.46}$$

from which it follows that

$$\begin{aligned}
\boldsymbol{B}^n(\boldsymbol{r}) &= \nabla \times \boldsymbol{A}^n(\boldsymbol{r}) = \int_S dS'\ B_n(\boldsymbol{r}')\nabla \times \boldsymbol{G}^n(\boldsymbol{r}, \boldsymbol{r}') \\
&= [1/(4\pi)] \int_S dS'\ B_n(\boldsymbol{r}')\nabla G(\mathbf{r}, \mathbf{r}') \\
&= -[1/(4\pi)] \int_S dS'\ B_n(\boldsymbol{r}')(\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|^3,
\end{aligned} \tag{19.3.47}$$

in accord with (3.9). Recall the vector identity

$$\nabla \times (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{D} \cdot \nabla)\boldsymbol{C} + \boldsymbol{C}(\nabla \cdot \boldsymbol{D}) - (\boldsymbol{C} \cdot \nabla)\boldsymbol{D} - \boldsymbol{D}(\nabla \cdot \boldsymbol{C}). \tag{19.3.48}$$

Using (3.31) and (3.48), show that

$$\nabla \times \boldsymbol{G}^t(\boldsymbol{r}, \boldsymbol{r}') = -[1/(4\pi)]\boldsymbol{n}'(\boldsymbol{r}')/|\boldsymbol{r} - \boldsymbol{r}'|^3 + [3/(4\pi)][\boldsymbol{n}'(\boldsymbol{r}') \cdot (\boldsymbol{r} - \boldsymbol{r}')](\boldsymbol{r} - \boldsymbol{r}')/|\boldsymbol{r} - \boldsymbol{r}'|^5, \tag{19.3.49}$$

from which it follows that

$$\begin{aligned}
\boldsymbol{B}^t(\boldsymbol{r}) &= \nabla \times \boldsymbol{A}^t(\boldsymbol{r}) = \int_S dS'\ \psi(\boldsymbol{r}')\nabla \times \boldsymbol{G}^t(\boldsymbol{r}, \boldsymbol{r}') \\
&= -[1/(4\pi)] \int_S dS'\ \psi(\boldsymbol{r}')\boldsymbol{n}'(\mathbf{r}')/|\mathbf{r} - \mathbf{r}'|^3 \\
&\quad + [3/(4\pi)] \int_S dS'\ \psi(\boldsymbol{r}')[\boldsymbol{n}'(\boldsymbol{r}') \cdot (\boldsymbol{r} - \boldsymbol{r}')](\boldsymbol{r} - \boldsymbol{r}')/|\boldsymbol{r} - \boldsymbol{r}'|^5.
\end{aligned} \tag{19.3.50}$$

Observe that, if we wish, we may define kernels $\boldsymbol{K}^n(\boldsymbol{r},\boldsymbol{r}')$ and $\boldsymbol{K}^t(\boldsymbol{r},\boldsymbol{r}')$ by the rules

$$\begin{aligned}\boldsymbol{K}^n(\boldsymbol{r},\boldsymbol{r}') &= \nabla\times\boldsymbol{G}^n(\mathbf{r},\mathbf{r}') = [1/(4\pi)]\nabla G(\mathbf{r},\mathbf{r}')\\ &= -[1/(4\pi)](\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|^3 \end{aligned} \tag{19.3.51}$$

and

$$\begin{aligned}\boldsymbol{K}^t(\boldsymbol{r},\boldsymbol{r}') &= \nabla\times\boldsymbol{G}^t(\boldsymbol{r},\boldsymbol{r}')\\ &= -[1/(4\pi)]\boldsymbol{n}'(\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|^3 + [3/(4\pi)][\boldsymbol{n}'(\boldsymbol{r}')\cdot(\boldsymbol{r}-\boldsymbol{r}')](\boldsymbol{r}-\boldsymbol{r}')/|\boldsymbol{r}-\boldsymbol{r}'|^5. \end{aligned} \tag{19.3.52}$$

With the aid of these definitions, (3.47) and (3.50) take the form

$$\boldsymbol{B}^n(\boldsymbol{r}) = \int_S dS'\ B_n(\boldsymbol{r}')\boldsymbol{K}^n(\boldsymbol{r},\boldsymbol{r}') \tag{19.3.53}$$

and

$$\boldsymbol{B}^t(\boldsymbol{r}) = \int_S dS'\ \psi(\boldsymbol{r}')\boldsymbol{K}^t(\boldsymbol{r},\boldsymbol{r}'). \tag{19.3.54}$$

Finally, write

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}^n(\boldsymbol{r}) + \boldsymbol{B}^t(\boldsymbol{r}). \tag{19.3.55}$$

Show that, for fixed $\boldsymbol{r}'$, $\boldsymbol{K}^n(\boldsymbol{r},\boldsymbol{r}')$ falls off as $1/r^2$ for large r and $\boldsymbol{K}^t(\boldsymbol{r},\boldsymbol{r}')$ falls off as $1/r^3$.

19.3.4. Show that the Cartesian components of $\boldsymbol{A}(\boldsymbol{r})$, as given by (3.5), (3.17), and (3.32), are harmonic functions,

$$\nabla^2\boldsymbol{A}(\boldsymbol{r}) = 0. \tag{19.3.56}$$

19.4 Numerical Benchmark

How well does the surface method described by relations (1.3) through (1.11) of Section 19.1 work in practice? The purpose of this subsection is to apply it to the monopole doublet test case of Sections 13.7, Chapter 16, and Chapter 17. In subsection 19.4.1 we will set up the Hamiltonian for particle motion in the field of a magnetic monopole doublet, select a particular design orbit, and determine a suitable bent box that contains this orbit as in Figure 1.1. In subsection 19.4.2 we will

19.4.1 Exact Field, Design Orbit Selection, and Choice of Surrounding Bent Box

Consider the monopole doublet magnetic field described by Equations (13.7.1) through (13.76) and Figures 13.7.1 through 13.7.5 of Section 13.7. As before, we will assign the values

$$a = 2.5 \text{ cm} = .025 \text{ m} \tag{19.4.1}$$

and

$$g = 1 \text{ Tesla (cm)}^2 = 1 \times 10^{-4} \text{ Tesla m}^2. \tag{19.4.2}$$

In order to set up the Hamiltonian that will describe particle motion in this field, we also need a vector potential $\boldsymbol{A}(\boldsymbol{r})$ such that

$$\nabla \times \boldsymbol{A}(\boldsymbol{r}) = \nabla\psi(\boldsymbol{r}) \tag{19.4.3}$$

with ψ given by (13.7.3). We will take the vector potential to be that of two Dirac magnetic monopoles of opposite sign. The first, with strength $\bar{g}$, will be situated at $\boldsymbol{r}^+ = a\boldsymbol{e}_y$ and will have a string extending from $\boldsymbol{r}^+$ to infinity along the positive y axis. The second, with strength $-\bar{g}$, will be situated at $\boldsymbol{r}^- = -a\boldsymbol{e}_y$ and will have a string extending from $\boldsymbol{r}^-$ to infinity along the negative y axis. See (2.63) and Figure 4.1. Thus, $\boldsymbol{A}(\boldsymbol{r})$ will be given by the relation

$$\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{A}^+(\boldsymbol{r}) + \boldsymbol{A}^-(\boldsymbol{r}) \tag{19.4.4}$$

with

$$\begin{aligned}
\boldsymbol{A}^+(\boldsymbol{r}) &= -\boldsymbol{A}_s(\boldsymbol{r}, \boldsymbol{r}^+, \boldsymbol{e}_y) \\
&= -[\bar{g}/(4\pi)][\boldsymbol{e}_y \times (\boldsymbol{r} - a\boldsymbol{e}_y)]/\{|\boldsymbol{r} - a\boldsymbol{e}_y|[|\boldsymbol{r} - a\boldsymbol{e}_y| - \boldsymbol{e}_y \cdot (\boldsymbol{r} - a\boldsymbol{e}_y)]\} \\
&= -[\bar{g}/(4\pi)](\boldsymbol{e}_y \times \boldsymbol{r})/\{|\boldsymbol{r} - a\boldsymbol{e}_y|[|\boldsymbol{r} - a\boldsymbol{e}_y| - y + a]\},
\end{aligned} \tag{19.4.5}$$

and

$$\begin{aligned}
\boldsymbol{A}^-(\boldsymbol{r}) &= -(-1)\boldsymbol{A}_s(\boldsymbol{r}, \boldsymbol{r}^-, -\boldsymbol{e}_y) \\
&= -[-\bar{g}/(4\pi)][-\boldsymbol{e}_y \times (\boldsymbol{r} + a\boldsymbol{e}_y)]/\{|\boldsymbol{r} + a\boldsymbol{e}_y|[|\boldsymbol{r} + a\boldsymbol{e}_y| + \boldsymbol{e}_y \cdot (\boldsymbol{r} + a\boldsymbol{e}_y)]\} \\
&= -[\bar{g}/(4\pi)](\boldsymbol{e}_y \times \boldsymbol{r})/\{|\boldsymbol{r} + a\boldsymbol{e}_y|[|\boldsymbol{r} + a\boldsymbol{e}_y| + y + a]\}.
\end{aligned} \tag{19.4.6}$$

Here, to compensate a pesky solid-angle factor of 4π that has crept in somewhere between the beginning of Section 13.7 and the end of Section 19.2.2, we have introduced the modified pole strength $\bar{g}$ defined by the relation

$$\bar{g} = 4\pi g. \tag{19.4.7}$$

Note that

$$\boldsymbol{e}_y \times \boldsymbol{r} = -x\boldsymbol{e}_z + z\boldsymbol{e}_x. \tag{19.4.8}$$

Therefore, in terms of components and taking (4.7) into account, the relation (4.4) takes the explicit form

$$\begin{aligned}
A_x(x, y, z) = &-\frac{gz}{[x^2 + (y-a)^2 + z^2]^{1/2}\{[x^2 + (y-a)^2 + z^2]^{1/2} - y + a\}} \\
&-\frac{gz}{[x^2 + (y+a)^2 + z^2]^{1/2}\{[x^2 + (y+a)^2 + z^2]^{1/2} + y + a\}},
\end{aligned} \tag{19.4.9}$$

$$A_y(x, y, z) = 0, \tag{19.4.10}$$

$$\begin{aligned} A_z(x,y,z) &= \frac{gx}{[x^2+(y-a)^2+z^2]^{1/2}\{[x^2+(y-a)^2+z^2]^{1/2}-y+a\}} \\ &\quad +\frac{gx}{[x^2+(y+a)^2+z^2]^{1/2}\{[x^2+(y+a)^2+z^2]^{1/2}+y+a\}}. \end{aligned} \tag{19.4.11}$$

From (2.28) and (4.9) through (4.11), and with some algebraic effort, it can be checked that

$$\begin{aligned} B_x &= \partial_y A_z - \partial_z A_y = \partial_y A_z = \\ &= gx[x^2+(y-a)^2+z^2]^{-3/2} - gx[x^2+(y+a)^2+z^2]^{-3/2}, \end{aligned} \tag{19.4.12}$$

$$\begin{aligned} B_y &= \partial_z A_x - \partial_x A_z \\ &= g(y-a)\{[x^2+(y-a)^2+z^2]^{-3/2} - g(y+a)[x^2+(y+a)^2+z^2]^{-3/2}, \end{aligned} \tag{19.4.13}$$

$$\begin{aligned} B_z &= \partial_x A_y - \partial_y A_x = -\partial_y A_x = \\ &= gz[x^2+(y-a)^2+z^2]^{-3/2} - gz[x^2+(y+a)^2+z^2]^{-3/2}, \end{aligned} \tag{19.4.14}$$

in agreement with (13.7.4) through (13.7.6). See Exercise 4.2.

We also note, for future use, that examination of (4.9) through (4.11) reveals that $\boldsymbol{A}(x,y,z)$ is even in y,

$$\boldsymbol{A}(x,-y,z) = \boldsymbol{A}(x,y,z) \tag{19.4.15}$$

and therefore

$$\boldsymbol{A}(x,y,z) = \boldsymbol{A}(x,0,z) + O(y^2). \tag{19.4.16}$$

To compute orbits (and maps) it is convenient to use z as the independent variable. In this case, and for the vector potential given by (4.9) through (4.11), the Hamiltonian becomes

$$K = -[p_t^2/c^2 - m^2c^2 - (p_x - qA_x)^2 - p_y^2]^{1/2} - qA_z. \tag{19.4.17}$$

See (1.6.16). Let β and γ be the usual relativistic factors defined by

$$\beta = v/c, \tag{19.4.18}$$

$$\gamma = (1-\beta^2)^{-1/2} \tag{19.4.19}$$

where v is the particle velocity. Then the magnitude of the mechanical momentum is given by the relation

$$p = \gamma m v = \gamma\beta m c \tag{19.4.20}$$

and the quantity p_t has the value

$$p_t = -(m^2c^4 + p^2c^2)^{1/2} = -\gamma m c^2. \tag{19.4.21}$$

Since K is independent of t, the quantities p_t and p will be constants of motion. Finally, let p^0 be the momentum for the design orbit.

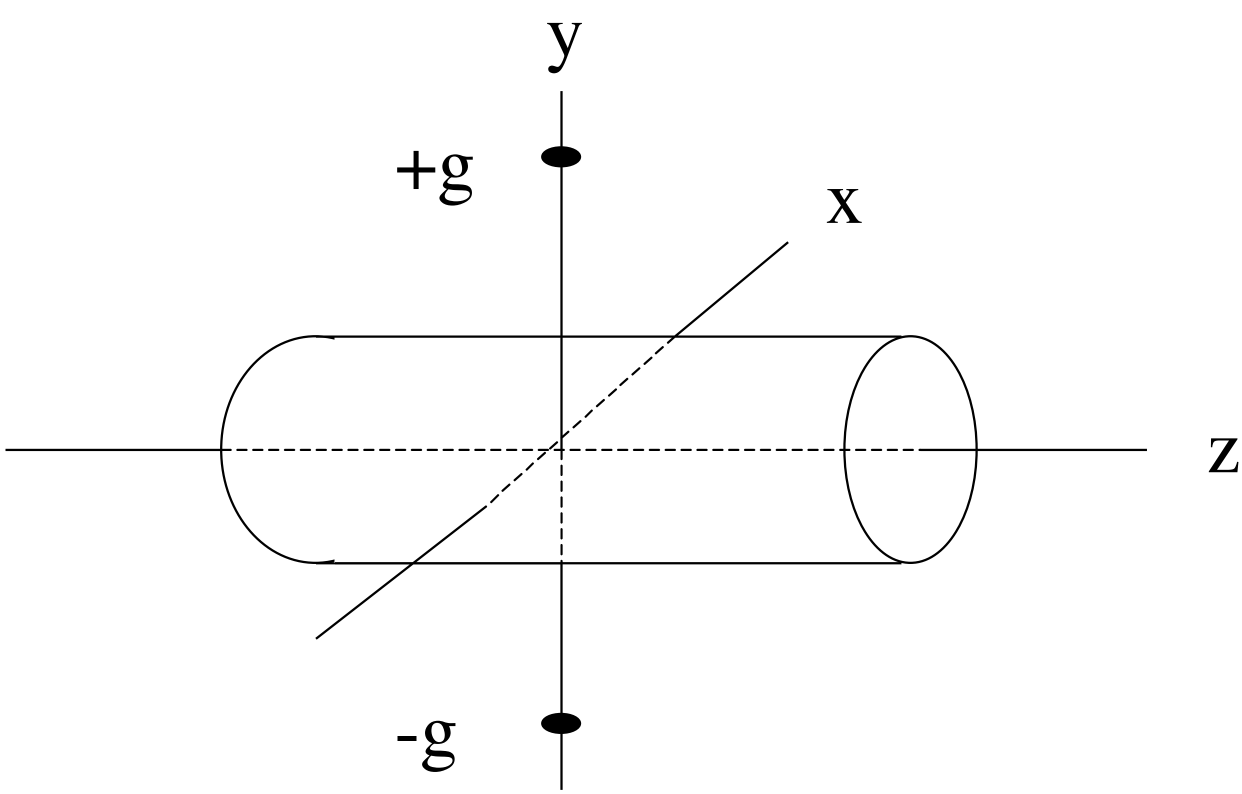

Figure 19.4.1: (Place holder) A monopole doublet consisting of two magnetic monopoles of equal and opposite sign placed on the y axis and centered on the origin. Also shown are the half-infinite Dirac strings extending from the $+\bar{g}$ monopole along the positive y axis and from the $-\bar{g}$ monopole along the negative y axis.

At this point it is useful to introduce dimensionless variables by the rules

$$X = x/\ell, \tag{19.4.22}$$

$$Y = y/\ell, \tag{19.4.23}$$

$$\tau = ct/\ell, \tag{19.4.24}$$

$$P_x = p_x/p^0, \tag{19.4.25}$$

$$P_y = p_y/p^0, \tag{19.4.26}$$

$$P_\tau = p_t/(p^0 c). \tag{19.4.27}$$

Here ℓ is a convenient scale length, and is not to be confused with the path length introduced in Exercise 1.7.6. The dimensionless variables satisfy the Poisson bracket rules

$$[X, P_x] = [Y, P_y] = [\tau, P_\tau] = 1/(\ell p^0). \tag{19.4.28}$$

From now on we will redefine their Poisson brackets so that conjugate variables again have unity Poisson brackets. This is permissible providing the Hamiltonian K is replaced by a properly scaled new Hamiltonian Hamiltonian H given by the relation

$$\begin{aligned} H &= -[1/(\ell p^0)]\{[(p^0 c)^2 P_\tau^2/c^2 - m^2c^2 - (p^0 P_x - qA_x)^2 - (p^0)^2 P_y^2]^{1/2} + qA_z\} \\ &= -(1/\ell)\{P_\tau^2 - (mc/p^0)^2 - (P_x - \mathcal{A}_x)^2 - P_y^2]^{1/2} + \mathcal{A}_z\} \end{aligned} \tag{19.4.29}$$

where

$$\mathcal{A}_x(X, Y, z) = (q/p^0)A_x(\ell X, \ell Y, z), \tag{19.4.30}$$

$$\mathcal{A}_z(X, Y, z) = (q/p^0)A_z(\ell X, \ell Y, z). \tag{19.4.31}$$

(See Appendix D.)

How should we chose a design trajectory? We would like it lie in the $y = 0$ plane, to pass through the origin, and to be symmetric about $z = 0$. That it is possible for there to be an orbit that lies in the $y = 0$ plane follows from (4.16). See Exercise 4.2. Also, observe from (4.9) through (4.11) that $\boldsymbol{A}(\boldsymbol{r})$ vanishes at the origin,

$$\boldsymbol{A}(0, 0, 0) = 0. \tag{19.4.32}$$

Therefore, the canonical and mechanical momenta agree at the origin. See (1.5.29). Consequently, and by symmetry, one way to achieve the desired design trajectory is to select, for $z = 0$, the initial conditions

$$X = Y = \tau = 0, \tag{19.4.33}$$

$$P_x = P_y = 0, \tag{19.4.34}$$

and then integrate both backward and forward in z to obtain the complete trajectory.

What remains is to select the values of P_τ and p^0. From (4.20) we see that for the design trajectory there is the relation

$$p^0 = \gamma^0 \beta^0 mc. \tag{19.4.35}$$

From (4.21) we see that the energy on this trajectory will be given by

$$p_t^0 = -\gamma^0 mc^2. \tag{19.4.36}$$

Therefore, on this trajectory P_τ has the value

$$P_\tau = P_\tau^0 = p_t^0/(p^0 c) = -\gamma^0 mc^2/(\gamma^0\beta^0 mcc) = -1/\beta^0. \tag{19.4.37}$$

And, with regard to the ingredients in (4.29), we see that

$$(P_\tau^0)^2 - (mc/p^0)^2 = (1/\beta^0)^2 - [1/(\gamma^0\beta^0)^2] = 1. \tag{19.4.38}$$

Therefore, on the design trajectory, H becomes

$$H = -(1/\ell)\{[1 - (P_x - \mathcal{A}_x)^2 - P_y^2]^{1/2} + \mathcal{A}_z\}. \tag{19.4.39}$$

[Note that, as it should, (4.39) agrees with (1.7.57) when $\delta = 0$.] Finally, we should select (by trial and error) the quantity p^0, which now appears only in (4.30) and (4.31), in such a way that, for the specified values of a and g, the design trajectory has some desired bend angle ϕ_{bend}. For purposes of illustration, we will require that ϕ_{bend} for a positron be approximately 30°.

Let us work out the equations of motion associated with H as given by (4.39). For convenience we will take the scale length to have the value

$$\ell = 1 \text{ m}. \tag{19.4.40}$$

We find the results

$$X' = \partial H/\partial P_x = (P_x - \mathcal{A}_x)/[1 - (P_x - \mathcal{A}_x)^2 - P_y^2]^{1/2}, \tag{19.4.41}$$

$$Y' = \partial H/\partial P_y = P_y/[1 - (P_x - \mathcal{A}_x)^2 - P_y^2]^{1/2}, \tag{19.4.42}$$

$$P_x' = -\partial H/\partial X = (\partial\mathcal{A}_x/\partial X)(P_x - \mathcal{A}_x)/[1 - (P_x - \mathcal{A}_x)^2 - P_y^2]^{1/2} + (\partial\mathcal{A}_z/\partial X), \tag{19.4.43}$$

$$P_y' = -\partial H/\partial Y = (\partial\mathcal{A}_x/\partial Y)(P_x - \mathcal{A}_x)/[1 - (P_x - \mathcal{A}_x)^2 - P_y^2]^{1/2} + (\partial\mathcal{A}_z/\partial Y). \tag{19.4.44}$$

Here a prime denotes d/dz. Also, from (4.9) and (4.11), we have the results

$$\begin{aligned}
\partial_x A_x \quad = \quad & \frac{gxz}{[x^2+(y-a)^2+z^2]\{a-y+[x^2+(y-a)^2+z^2]^{1/2}\}^2} \\
& + \frac{gxz}{[x^2+(y-a)^2+z^2]^{3/2}\{a-y+[x^2+(y-a)^2+z^2]^{1/2}\}} \\
& + \frac{gxz}{[x^2+(y+a)^2+z^2]\{a+y+[x^2+(y+a)^2+z^2]^{1/2}\}^2} \\
& + \frac{gxz}{[x^2+(y+a)^2+z^2]^{3/2}\{a+y+[x^2+(y+a)^2+z^2]^{1/2}\}},
\end{aligned} \tag{19.4.45}$$

$$
\begin{aligned}
\partial_x A_z \quad = \quad & \frac{-gx^2}{[x^2+(y-a)^2+z^2]\{a-y+[x^2+(y-a)^2+z^2]^{1/2}\}^2} \\
& - \frac{gx^2}{[x^2+(y-a)^2+z^2]^{3/2}\{a-y+[x^2+(y-a)^2+z^2]^{1/2}\}} \\
& + \frac{g}{[x^2+(y-a)^2+z^2]^{1/2}\{a-y+[x^2+(y-a)^2+z^2]^{1/2}\}} \\
& - \frac{gx^2}{[x^2+(y+a)^2+z^2]\{a+y+[x^2+(y+a)^2+z^2]^{1/2}\}^2} \\
& - \frac{gx^2}{[x^2+(y+a)^2+z^2]^{3/2}\{a+y+[x^2+(y+a)^2+z^2]^{1/2}\}} \\
& + \frac{g}{[x^2+(y+a)^2+z^2]^{1/2}\{a+y+[x^2+(y+a)^2+z^2]^{1/2}\}}, \qquad (19.4.46)
\end{aligned}
$$

$$\partial_y A_x = \frac{-gz}{[x^2+(y-a)^2+z^2]^{3/2}} + \frac{gz}{[x^2+(y+a)^2+z^2]^{3/2}}, \qquad (19.4.47)$$

$$\partial_y A_z = \frac{gx}{[x^2+(y-a)^2+z^2]^{3/2}} - \frac{gx}{[x^2+(y+a)^2+z^2]^{3/2}}. \qquad (19.4.48)$$

Together, these relations provide the equations of motion.

Upon integrating these equations of motion, with the initial conditions (4.33) and (4.34), we find, as possible precise values, the combination

$$\phi_{\text{bend}} =, \qquad (19.4.49)$$

when

$$qg/p^0 = *. \qquad (19.4.50)$$

Correspondingly, we find the values

$$p^0 =, \qquad (19.4.51)$$

$$\beta^0 =, \qquad (19.4.52)$$

$$\gamma^0 = . \qquad (19.4.53)$$

Figure 4.2 shows the spatial part of the design trajectory found in this way. Figure 4.3 displays $P_x(z)$ on this trajectory. (Of course, we also have $y = 0$ and $P_y = 0$ on this trajectory.) Also shown in Figure 4.2 is the top view of a suitable bent box that surrounds this trajectory. The top and bottom of the box lie in the planes $y = \pm *$ cm. The circular arcs have the common center

$$(x_c, z_c) = (*\text{cm}, 0) \qquad (19.4.54)$$

and have radii

$$r_{\text{out}} = *\text{cm}, \qquad (19.4.55)$$

$$r_{\text{in}} = *\text{cm}. \qquad (19.4.56)$$

The straight ends of the box, not fully shown, have lengths of $*$cm.

It is also useful to have graphics of the quantities $\mathcal{A}_x$ and B_y along the design trajectory. They are displayed in Figures 4.4 and 4.5.

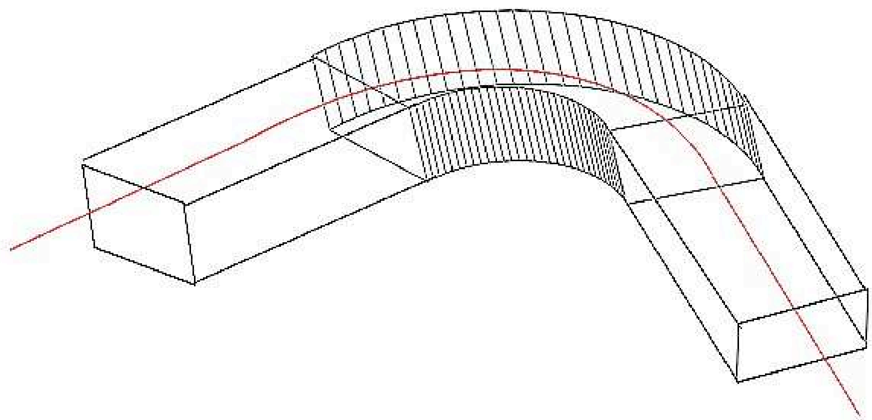

Figure 19.4.2: (Place holder)Design trajectory $X(z)$ and surrounding bent box.

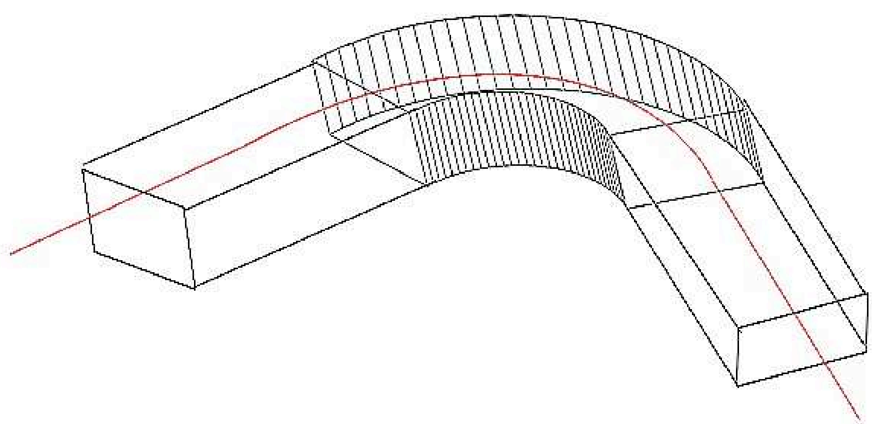

Figure 19.4.3: (Place holder)The momentum $P_x(z)$ on the design trajectory.

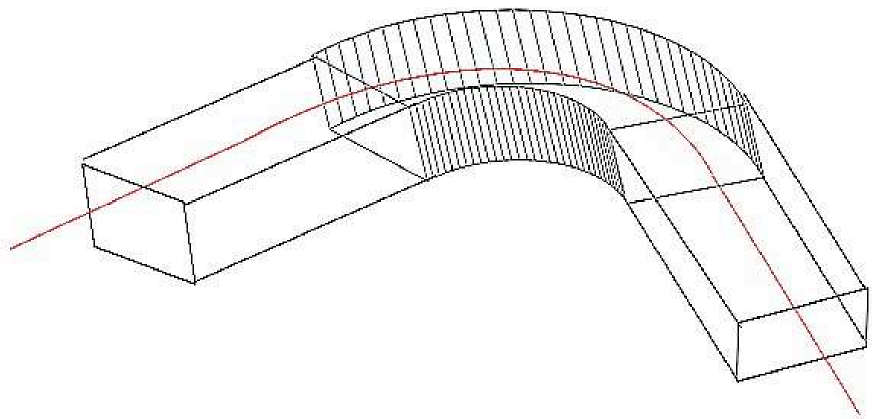

Figure 19.4.4: (Place holder)The quantity $\mathcal{A}_x$ along the design trajectory.

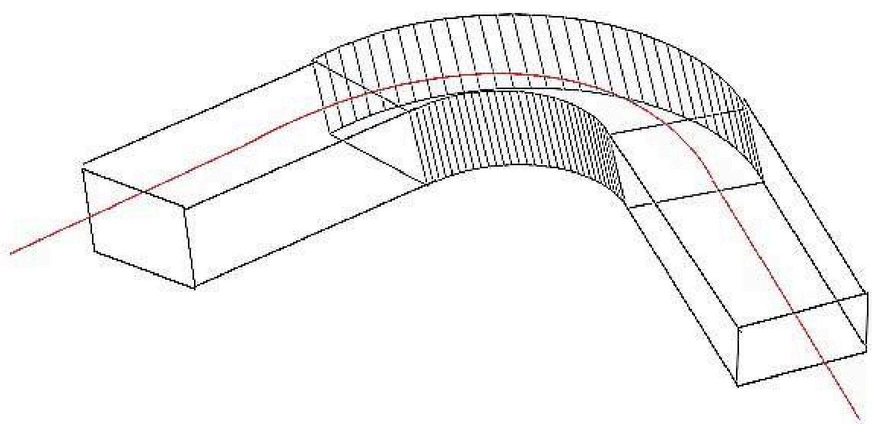

Figure 19.4.5: (Place holder)The quantity B_y along the design trajectory.

19.4.2 Bent Box Results

19.4.3 Comparison of Maps

Exercises

19.4.1. Verify (4.4) through (4.11).

19.4.2. Show that the components of $\boldsymbol{A}^+$ are given by the relations

$$A_x^+(x,y,z) = -\frac{gz}{[x^2+(y-a)^2+z^2]^{1/2}\{[x^2+(y-a)^2+z^2]^{1/2}-y+a\}}, \tag{19.4.57}$$

$$A_y^+(x,y,z) = 0, \tag{19.4.58}$$

$$A_z^+(x,y,z) = \frac{gx}{[x^2+(y-a)^2+z^2]^{1/2}\{[x^2+(y-a)^2+z^2]^{1/2}-y+a\}}. \tag{19.4.59}$$

Verify that

$$\nabla \times \boldsymbol{A}^+(\boldsymbol{r}) = g(\boldsymbol{r}-a\boldsymbol{e}_y)/|\boldsymbol{r}-a\boldsymbol{e}_y|^3. \tag{19.4.60}$$

Infer an analogous result for $\boldsymbol{A}^-$, and hence verify (4.12) through (4.14).

19.4.3. Verify (4.15) and (4.16). Show that any trajectory having the initial conditions $y = 0$ and $P_y = 0$ must lie in the $y = 0$ plane.

19.5 Smoothing and Insensitivity to Errors

19.6 Application to a Storage-Ring Dipole

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