End Field Modelling

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End Field Model

- Make some assumption on behavior of field at ends
  - Rate and form of falloff
  - Symmetry
- Types of end symmetry
  - Midplane: form of field in midplane is given: $B_y(x, 0, s)$
  - Multipole: in polar coordinates, B_r and B_ϕ in polar coordinates are of the form $f(r, s)\sin[(m+1)\phi]$ (cos for the other)
    - ⋆ Specify coefficient of $r^m \sin[(m+1)\phi]$ (cos for the other)
- These assumptions give different answers
  - Answers are the same if there is no s dependence
  - Which symmetry to choose depends on magnet construction
  - Could be other symmetries

Example: Quadrupole

- Maintain multipole symmetry:

$$B_x = -\sum_{k=0} \frac{1}{2k!(k+2)!} B_1^{(2k)}(s)[(2k+1)x^2 y + y^3]\left(-\frac{x^2+y^2}{4}\right)^{k-1}$$

$$B_y = -\sum_{k=0} \frac{1}{2k!(k+2)!} B_1^{(2k)}(s)[x^3 + (2k+1)xy^2]\left(-\frac{x^2+y^2}{4}\right)^{k-1}$$

$$B_s = \sum_{k=0} \frac{1}{k!(k+2)!} B_1^{(2k+1)}(s)(x^2 - y^2)\left(-\frac{x^2+y^2}{4}\right)^{k}$$

- Midplane expansion

$$B_x = \sum_{k=0} \frac{1}{(2k+1)!}(-1)^k B_1^{(2k)}(s) y^{2k+1} \qquad B_y = x\sum_{k=0} \frac{1}{(2k)!}(-1)^k B_1^{(2k)}(s) y^{2k}$$

$$B_s = x\sum_{k=0} \frac{1}{(2k+1)!}(-1)^k B_1^{(2k+1)}(s) y^{2k+1}$$

Example: Quadrupole: Notes

- Very different behaviors
- Multipole is not linear in midplane
- Midplane expansion has higher multipole components
- Note midplane is always linear in x
  - similar true for higher multipoles, but only in straight coordinate system
- Fields are sum of terms
  - s-dependence of each coeffienent is some derivative of a given function
  - Will be true as long as curvatures are constant

Example: Midplane Expansion for Bend

- Given B_y in midplane
- Planar reference curve
- Want sufficient terms to get correct linear behavior
- Vector potentials

$$A_{s0}(x,s) = -\frac{1}{1+hx}\int_0^x (1+h\bar{x})B_{y0}(\bar{x},s)\,d\bar{x}$$
$$A_{y1}(x,s) = \frac{1}{(1+hx)^2}\int_0^x (1+h\bar{x})\partial_s B_{y0}(\bar{x},s)\,d\bar{x}$$
$$A_{x2}(x,s) = -\frac{2h}{(1+hx)^3}\int_0^x (1+h\bar{x})\partial_s B_{y0}(\bar{x},s)\,d\bar{x}$$
$$A_{s2}(x,s) = \partial_x B_{y0}(x,s) + \frac{1}{(1+hx)^3}\int_0^x (1+h\bar{x})\partial_s^2 B_{y0}(\bar{x},s)\,d\bar{x}.$$

Hard-Edge End Field Approximation

- This does not mean no end field!
- Attempt to extract maximum information without knowing details of end
- Want to examine multiple designs
- Can’t re-design magnets each time you make a lattice change
- Need good starting point to judge nonlinearities
  - Coming from end fields
  - Chromatic behavior
  - Dynamic aperture

Lie Algebra in One Slide

- Poisson Bracket $[f, g]$:

$$[f, g] = \sum_k \left(\frac{\partial f}{\partial x_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial x_k} \right)$$

- Lie operator f acting on g: $:f:g = [f, g]$
- Lie map $e^{:f:}$ acts on a function; in particular, acts on coordinate functions
  - Gives evolution of coordinates
  - Exponential form makes it exactly symplectic
  - Satisfies Hamilton's equations for Hamiltonian H:

$$\frac{d}{ds} e^{:f:} = -e^{:f:}{:}H{:}$$

Tracking Through Magnet Ends

- Compute result to first order in body field strength
  - Can be computed independent of end shape
  - Arbitrary order in transverse variables
  - Limit as end length goes to zero
  - Can’t do better than this without knowing end field shape
- Hamiltonian $H_p - H_q$
  - H_p independent of field
  - H_q linear in field
  - Other terms ignored in this approximation

Tracking Through Magnet Ends (cont.)

- Write map as $e^{:f_p(s):}e^{:f_q(s):}$, f_p independent of field, f_q linear in field

$$\frac{d}{ds}e^{:f_p(s):} = -e^{:f_p(s):}{:}H_p{:}$$

  - $e^{:f_p(s):}$ will become the identity map as end length $\to 0$.
  - Still needed as part of derivation

- Now have differential equation for f_q (need to know fancy Lie algebra stuff for this)

$$\mathrm{iex}(-{:}f_q{:})\frac{df_q}{ds} = H_q + (e^{-:f_q:} - 1)H_p \qquad \mathrm{iex}(x) = \frac{e^x - 1}{x}$$

- Write f_q as a sum of terms, and get recursion relation (ignore nonliner in f_q)

$$f_q(s) = \sum_{k=1} f_k(s)$$

$$f_1(s) = \int^s H_q(\bar{s})\,d\bar{s} \qquad f_{n+1}(s) = \int^s [H_p, f_n(\bar{s})]\,d\bar{s}$$

Tracking Through Magnet Ends (cont.)

- If $\mathcal{S}_L(s)$ is a function going from 0 to 1 in length L, $L \to 0$,

$$\int_{-L/2}^{L/2} ds_1 \int_{-L/2}^{s_1} ds_2 \cdots \int_{-L/2}^{s_{n-1}} ds_n\, \mathcal{S}_L^{(k)}(s_n) = \delta_{kn}$$

- Accelerator Hamiltonian with curvatures h_x and h_y:

$$[H_p, f] = -\left[h_x p_s \frac{\partial f}{\partial p_x} + h_y p_s \frac{\partial f}{\partial p_y} + (1 + h_x x + h_y y)\left(\frac{p_x}{p_s}\frac{\partial f}{\partial x} + \frac{p_y}{p_s}\frac{\partial f}{\partial y}\right)\right]$$

- Thus f_k picks off terms proportional to the kth derivative of the field at the end
  - Assumes reference curve curvatures are constant
- Result is that f_{n+1} has larger transverse order than f_n: convergence, in some sense
- Evaluation: only need to get correct to first order:
  $\boldsymbol{z}_{\text{new}} = \boldsymbol{z}_{\text{old}} + J\boldsymbol{\nabla} f\Big((\boldsymbol{z}_{\text{old}} + \boldsymbol{z}_{\text{new}})/2\Big)$, J is symplectic metric
  - Method is symplectic, but implicit: probably nothing better for symplectic
  - Can do Euler step if don't need symplecticity

Example: Bend

- Use midplane expansion from above
- Get linear effects correct

$$f = \frac{qy^2p_x}{2p_s}\Delta B_{y0}(x)$$

- If only looking to get tunes right:

$$\Delta p_y = -\frac{qyp_x}{p_s}\Delta B_{y0}(x)$$

- We could track with this, and would already see nonlinear behavior
  - Should probably include at least one higher order to get some pure y nonlinearity
- This is the classical result, but we have more
  - This works for arbitrary midplane field profile, everywhere in midplane, and gets linear behavior correct
  - We know how to treat the corresponding nonlinearities
  - We can expand to higher order

Conclusions

- When doing a field expansion, it is important to choose the correct symmetry
  - Symmetry corresponds to magnet construction
- Can get results from effects of magnet ends without knowing much about magnet ends
  - Still need to know general symmetry
  - Can get higher order nonlinearities: dynamic aperture