Longitudinal Motion in Nearly-Isochronous FFAGs

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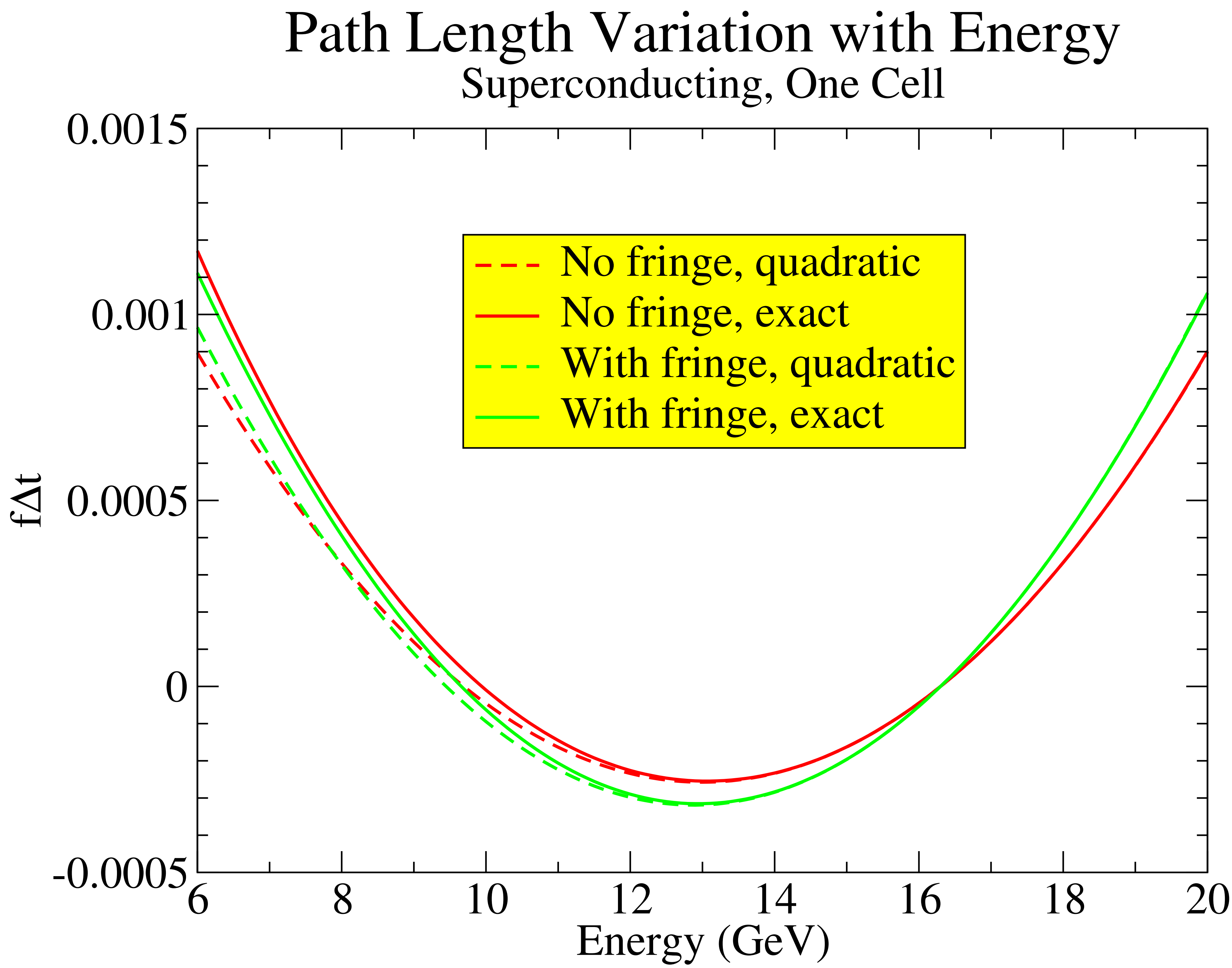
Introduction

- Path length in FFAG arc not independent of energy.
  - Very nearly quadratic
  - Particle won't come back to RF at same phase each time
  - High-frequency RF: walk off crest
- Given the maximum path length deviation, how many turns can a particle be accelerated
  - Answer: infinite
  - Linac voltage goes to nonzero value as turns go to infinity
  - Minimum voltage increases linearly with path length deviation
- How does acceptance vary with number of turns?



Path Length Variation with Energy
Superconducting, One Cell
0.0015
0.001
0.0005
0
-0.0005
fΔt
6
8
10
12
14
16
18
20
Energy (GeV)
No fringe, quadratic
No fringe, exact
With fringe, quadratic
With fringe, exact

Lattice Description

- Alternating sequence of cavities/linacs and arcs
- Two types of systems
  - Racetrack (or small number of sides): two long linacs, two long arcs
  - Distributed RF: many short arcs alternating with single cavities
- Path length varies quadratically with energy in arcs
  - Minimum path length at middle energy
  - Being exactly quadratic not essential
  - Can adjust total arc length so that zero (relative to integer number of RF cycles) occurs for any energy you want (different trick for distributed RF)
- All cavities/linacs have same phase
- Ignore time-of-flight change in linac

Equations

- Equations of motion

$$E_{n+1} = E_n + V\cos(\omega\tau_n) \qquad \tau_{n+1} = \tau_n + \Delta T\left(\frac{E_{n+1} - E_{\text{avg}}}{\Delta E/2}\right)^2 - T_0$$

  - E_n is energy after nth cavity/linac pass
  - τ_n is time-of-flight relative to crest in nth linac
  - ΔE is total energy gain
  - E_{avg} is middle energy
  - ΔT is the difference in time-of-flight from the minimum to the end of the parabola
  - T_0 a time-of-flight offset that can be generated by changing the arc length (or relative cavity phasing in a distributed RF system)

Finding System Parameters

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- Fix N, the number of turns, and $E_N - E_0 = \Delta E$, the total energy gain.
- ΔT is given by the lattice design
- Minimize V, the required RF voltage by varying
  - τ_0, the time at which you enter the first cavity/linac
  - T_0, the time-of-flight offset from zero of the minimum of the parabola
    - For long arcs (generally few long linacs per turn), adjust arc length slightly
    - For short arcs (many arcs and small cavities distributed evenly around the ring), adjust relative cavity phases
      - If M cavities per turn, relative phase a multiple of $2\pi/M$

Continuous Approximation

- Write discrete equations as differential equations

$$\frac{dE}{dn} = V\cos(\omega\tau) \qquad\qquad \frac{d\tau}{dn} = \Delta T\left(\frac{E - E_{\text{avg}}}{\Delta E/2}\right)^2 - T_0$$

- Eliminate n

$$\left[\Delta T\left(\frac{E - E_{\text{avg}}}{\Delta E/2}\right)^2 - T_0\right] dE = V\cos(\omega\tau)\,d\tau$$

Continuous Approximation (cont.)

- Integrate both sides

$$\frac{\Delta T \Delta E}{6}\left(\frac{E - E_{\text{avg}}}{\Delta E/2}\right)^3 + \frac{\Delta T \Delta E}{6} - T_0(E - E_0) = \frac{V}{\omega}[\sin(\omega\tau) - \sin(\omega\tau_0)]$$

  - RHS is bounded for fixed V
  - LHS has two internal extrema, plus value at maximum E.
  - Maximum of these three determines minimum V
  - Choose T_0 minimizing that maximum

- That optimum occurs when

$$T_0 = \frac{\Delta T}{4} \qquad \omega\tau_0 = \frac{\pi}{2} \qquad V = \frac{\omega \Delta T \Delta E}{24}$$

Note $V \propto (\Delta E)^3$ for given arcs

Description of Motion

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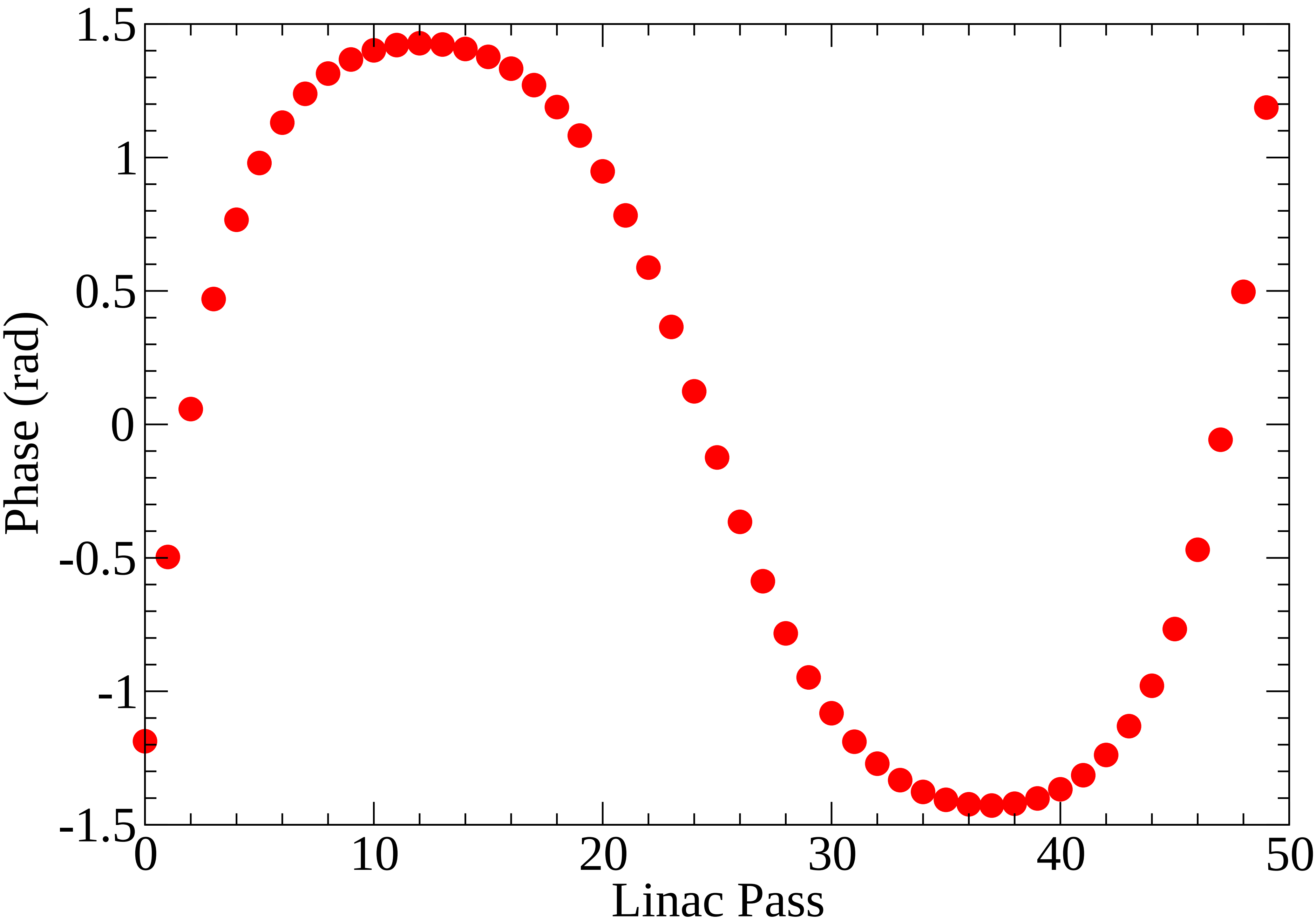
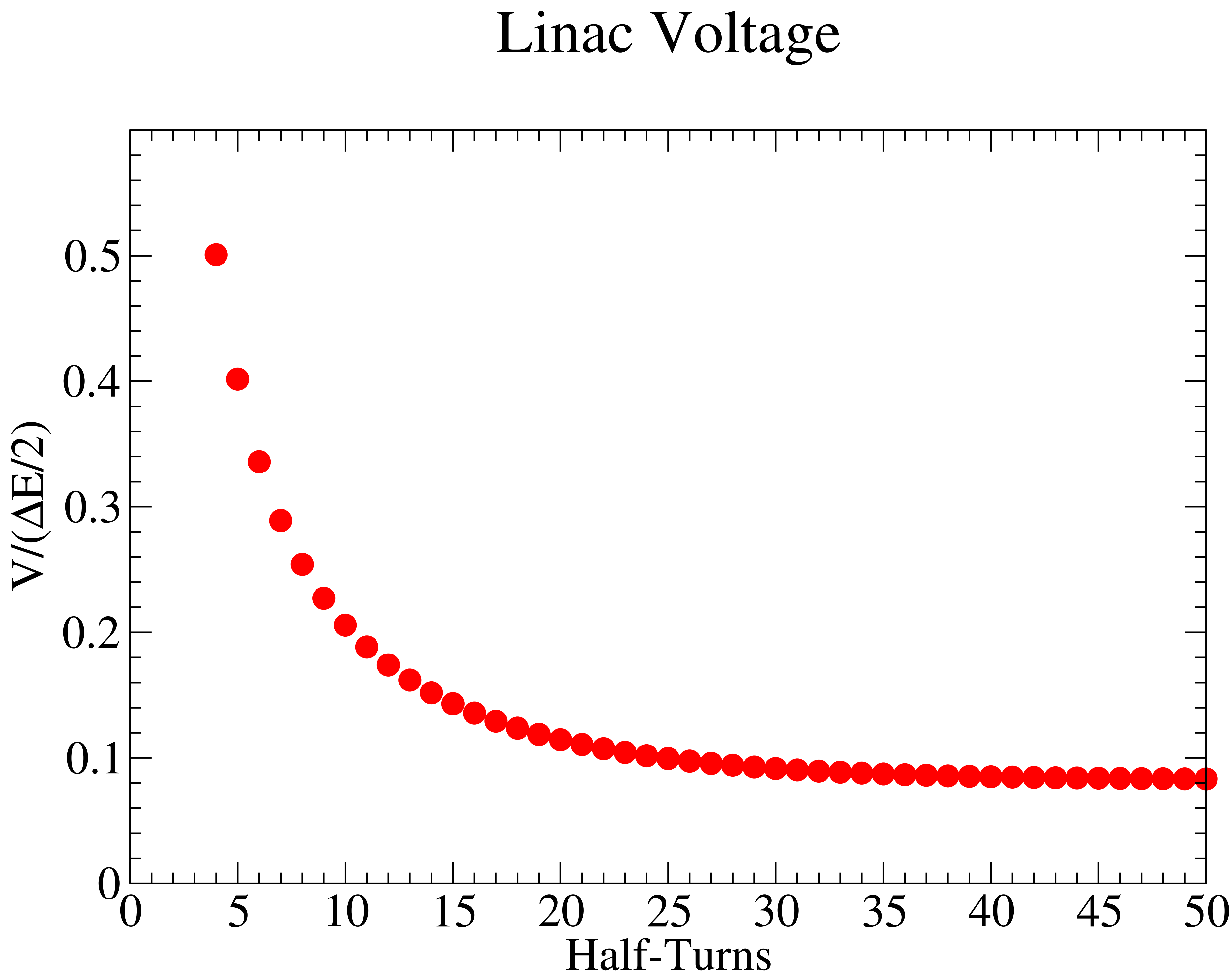
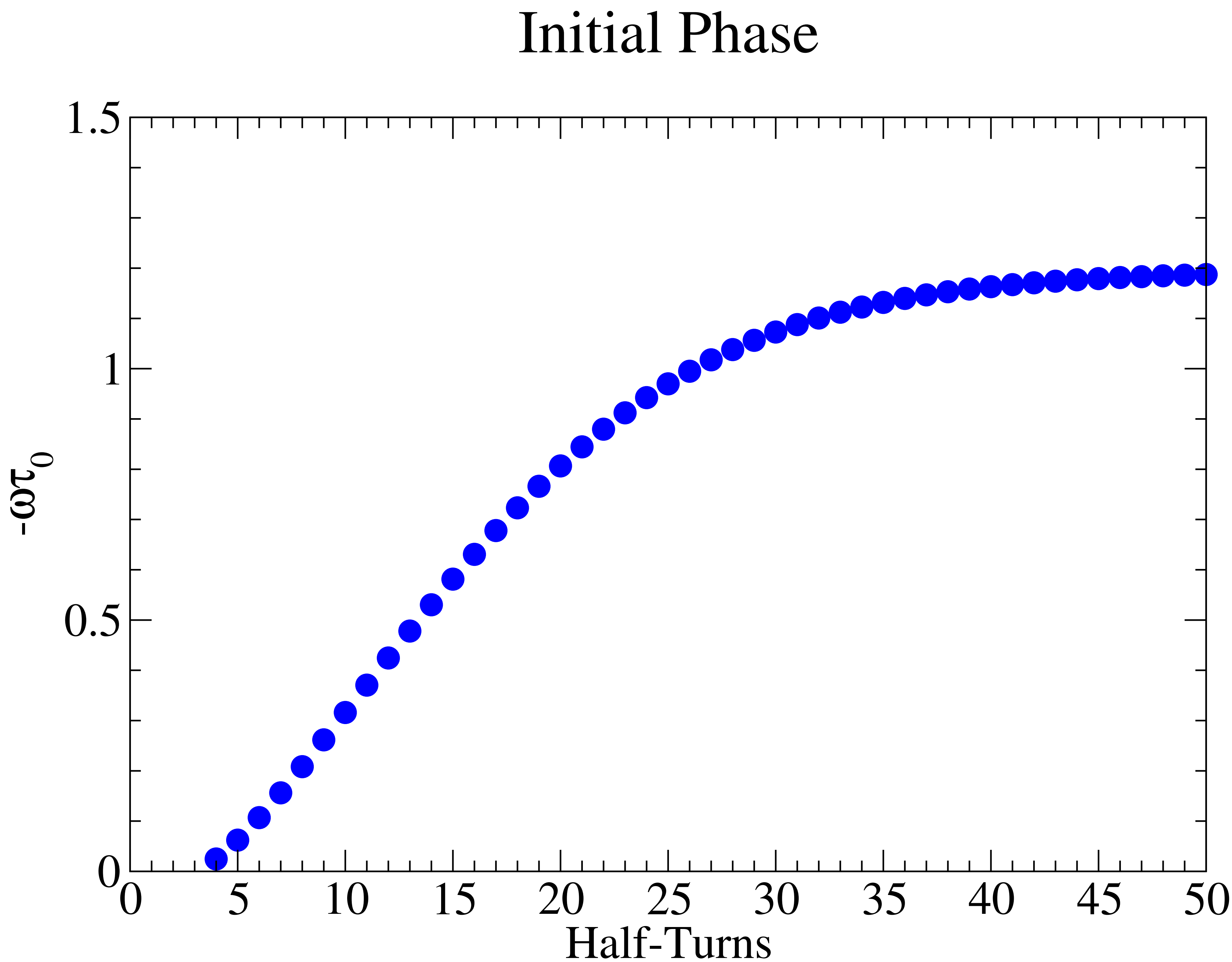
- Cross crest three times
  - Starts far off-crest, oscillates to about same distance off-crest on other side
- Validity of continuous approximation
  - As $N \to \infty$ with $N\Delta\phi$ fixed, solution approaches continuous approximation
    - ⋆ Distributed RF
  - If fix $\Delta\phi$ and let $N \to \infty$:
    - ⋆ Racetrack
    - ⋆ Appears that T_0 and V nearly continuous values
    - ⋆ Reason for differences: $\Delta\phi$ still gives a finite phase jump at beginning for large N
    - ⋆ Reason can still have infinite N: much time spent near synchronous phase


1.5
1
0.5
0
-0.5
-1
-1.5
Phase (rad)
0
10
20
30
40
50
Linac Pass

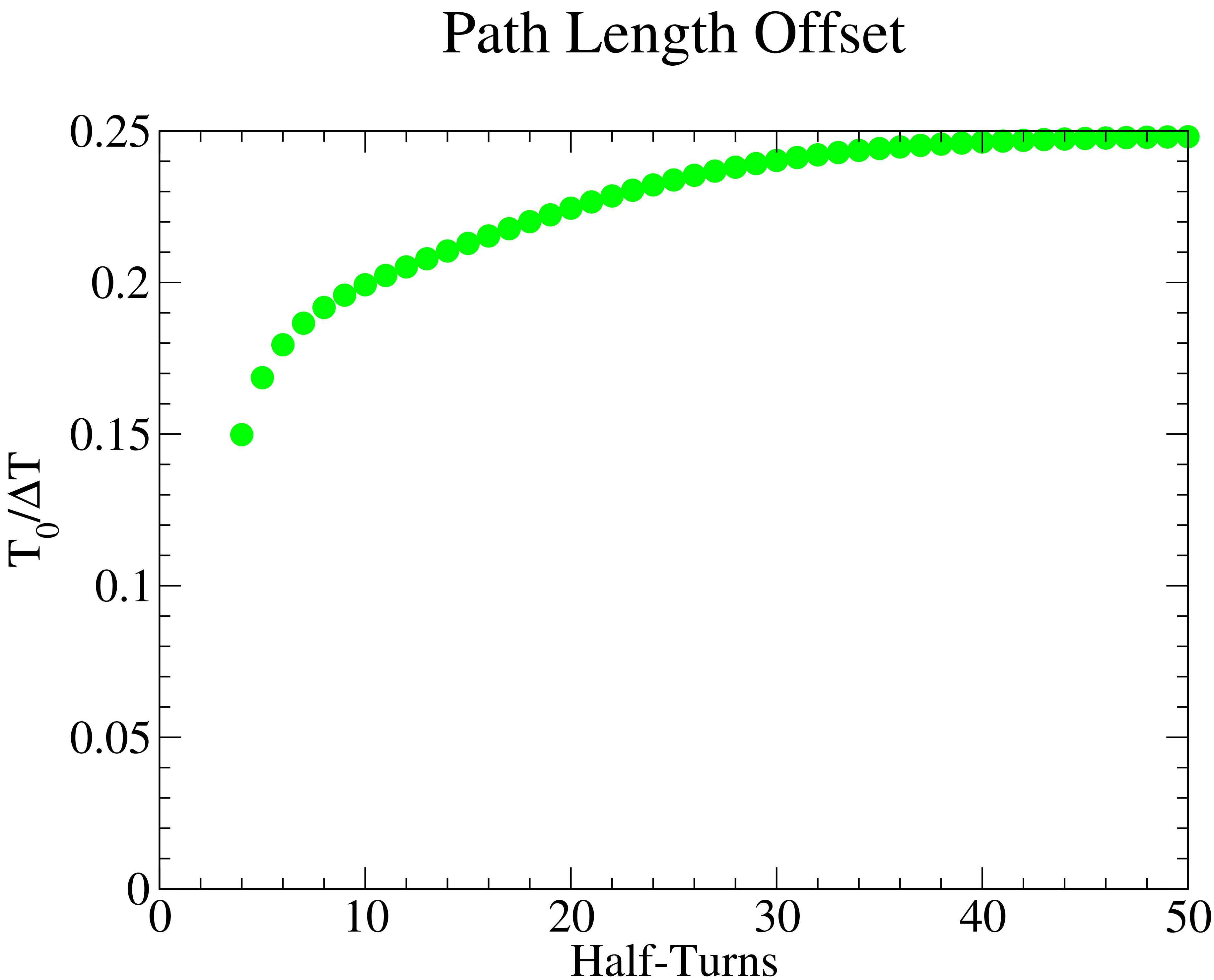
Sample Solution

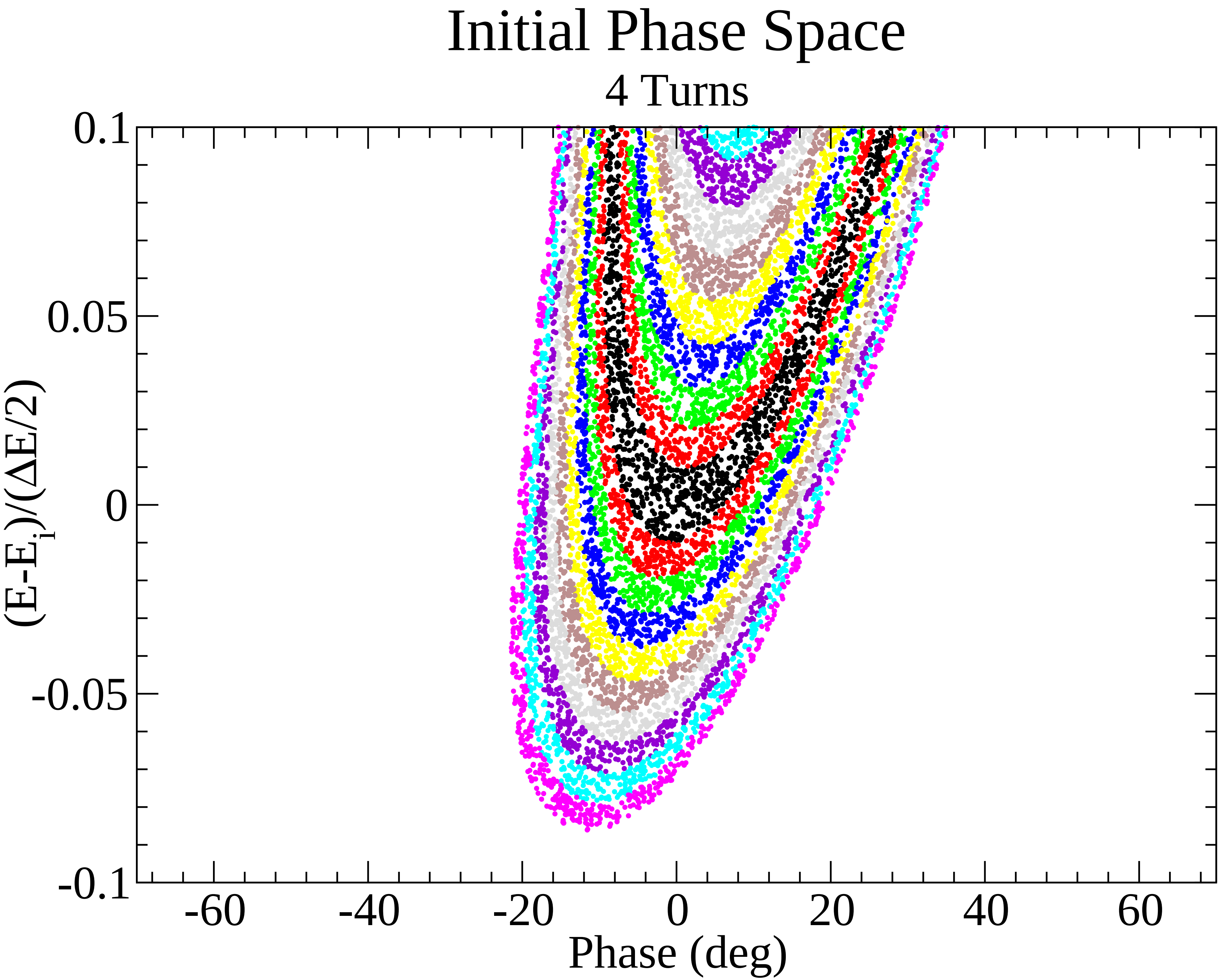
- $\Delta\phi = 1$ radian
- Can get large number of turns
- Little to be gained in cost by going past 30 half-turns
- Voltage and T_0 approach continuous approximation values
  - V actually a bit less
- Note large number of turns spent near phase extrema
- Acceptance
  - Appears to decrease with increasing turns
  - Width increases with increasing turns: crest-crossing

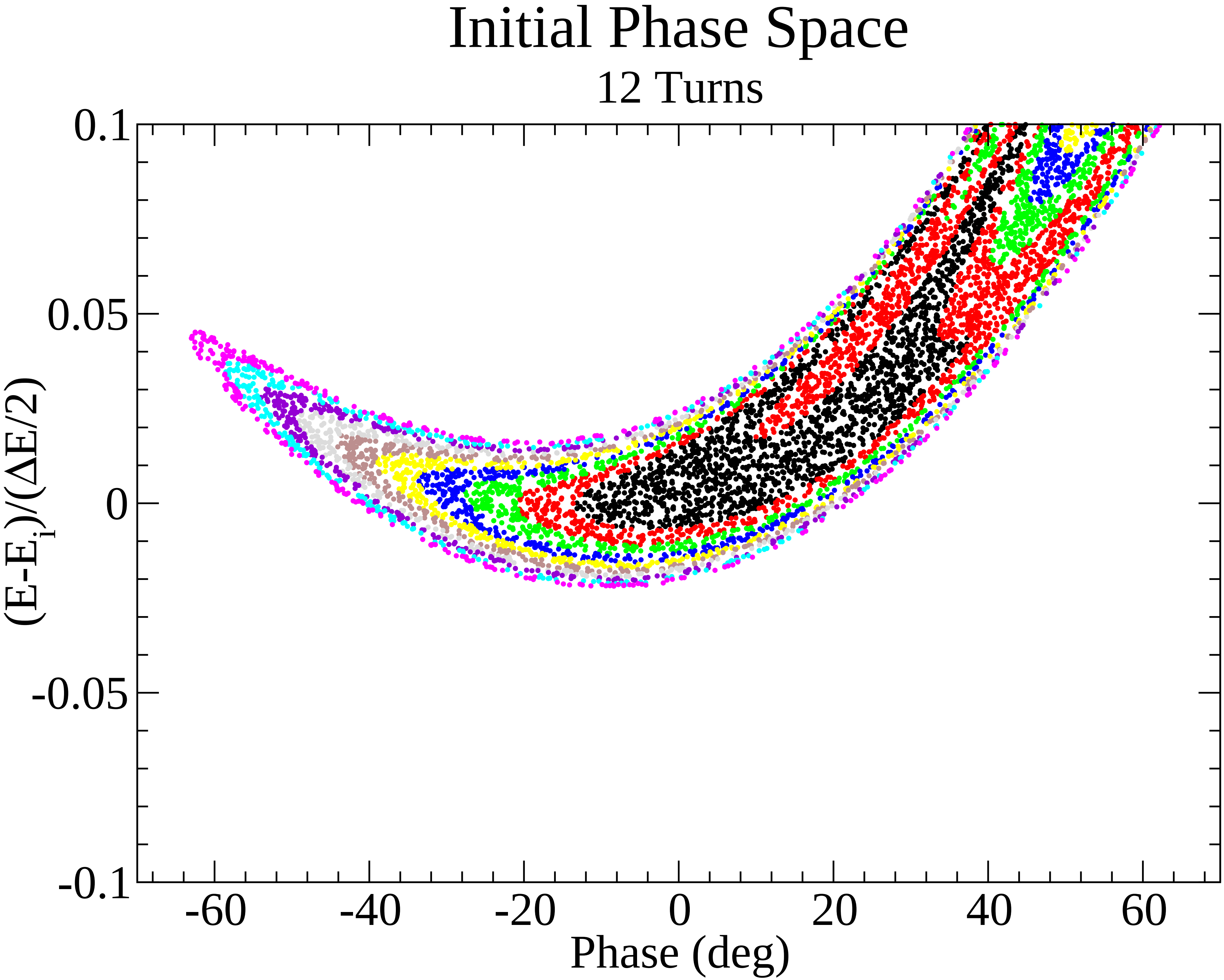

Linac Voltage
V/(ΔE/2)
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
20
25
30
35
40
45
50
Half-Turns



Initial Phase
1.5
1
0.5
0
-ωτ0
0
5
10
15
20
25
30
35
40
45
50
Half-Turns


Path Length Offset
0.25
0.2
0.15
0.1
0.05
0
T0/ΔT
0
10
20
30
40
50
Half-Turns


Initial Phase Space
4 Turns
0.1
0.05
0
-0.05
-0.1
(E-E_i)/(ΔE/2)
-60
-40
-20
0
20
40
60
Phase (deg)



Initial Phase Space
12 Turns
0.1
0.05
0
-0.05
-0.1
(E-E$_i$)/(ΔE/2)
-60
-40
-20
0
20
40
60
Phase (deg)

Conclusions and Improvements

- Can quickly compute minimum voltage needed in FFAG accelerator
- Can accelerate for arbitrary number of turns
  - Linac voltage does not go to zero
  - Phase space acceptance decreases
- Results for different voltage profile or time-of-flight profile qualitatively similar
  - Linear time-of-flight would cross crest twice. Less optimal for given total swing.
- Improvements
  - Allow linacs to have different phases, also different arcs
  - Increase voltage above optimum: better phase space acceptance?