FREE ELECTRON LASER THEORY USING TWO TIMES GREEN FUNCTION FORMALISM

HIROSHI TAKAHASHI

Brookhaven Natioanal Laboratory
Upton New York, 11973

In this paper, we present a quatum theory for free electron laser obtained by firstly using the Two time's Green Function method developed by Matsubara for solid physics theory. The dispersion relation for the laser photon obtained is limited to the case of low intensity of the laser due to the decoupling the correlation function in low order. For the analysis of the self-amplified emission (SASE), the high intensity laser radiation which strongly affect the trajectory of the free electron is involved, the use of the classical approximation for laser can formulate the laser radiation with multiple frequency. To get the quantum effects in the high intensity laser, use of the perturbation theory, and the expansion methods of state function using the coherent, squeeze and super-radiant states have discussed.

1. Introduction

The classical formalism for an electro-magnetic field has been used to formulate a theory for the free electron laser Because the electron beam used for the free electron laser was not very well refined in the past, its formulation by using quantum mechanical formalism was not required. However due to the recent refinements of the electron particle beam used for the free electron laser, it is necsessary to formulate a theory and to it give a good foundation by taking the quantum effect into account the Twenty years ago, I derived a quantum theory based on the two times Green function formalism derived by Matsubara for solid physics theory[1]. The dispersion relation is compared with the classical theory obtained by Kwan[2] .
When a relativistic electron moving in a electro-magnetic field and the magnetic field B , The Hamiltonian for such a system may be writteen as

$$H = \sum_i (c\, \alpha_i \cdot (p_i - eA/c) + \beta_i mc^2) + H_r + H_{ee} + H_s$$

$$H = \sum_i (c\, \alpha_i.(p_i - -eA/c) + \beta_i\, mc^2) + H_r + H_{ee} + H_s \qquad (1.1)$$

where A is the sum of the vector potential A_s of the B-field and the vector potential A_r of the electromagnetic wave; A $=A_s + A_r$, m and c are the mass of the electrons and the speed of light, respectively, while α and β are the Dirac matrix operators, the suffix stands for the ith particle. H_r is the

Hamiltonian for the electromagnetic wave. H_{ee} is the Hamiltonian for the interaction between electrons , and H_s is the Hamiltonian for the Static magnetic field. The last Hamiltonian, which is the constant of motion, is not incorporated in the dynamic of the system; it can thus be neglected in the following derivation.

The Hamiltonian H_{ee} and H_r are given by

$$H_{ee} = \sum_{.i.>j} e^2 / r^{ij}, (1.2)$$

$$H_{ee} = \sum_{i>j} e^2 / r^{ij}, \quad (1.2)$$

$$H_r = \sum_{q\lambda} h\, \omega_q\, b^+_{q,\lambda}\, b_{q,\lambda} \quad (1.3)$$

where $b^+_{q,\lambda}$ and $b_{q,\lambda}$ are the photon creation and annihilation operator, respectively, for photon momentum h q and polarization state λ. . The photon energy is given by $h\,\omega_q = hcq$. The unperturbed system is described by an electron moving in the B-field and the free radiation. Hamiltonian for the unperturbed system is

$$H_0 = H_e + H_r$$

$$H_0 = H_e + H_r \qquad (1.4)$$

where

$$H = \sum_{q\lambda} h\omega_q b^+_{q,\lambda} b_{q,\lambda}$$

$$H_e = \sum_i (c\, \alpha_i (p_i - -eA_{si} / c) + \beta_i mc^2) \qquad (1.5)$$

The interaction Hamiltonian is expressed as

$$H_{int} = - \sum_i e\, \alpha_i A_{ri} +\_H_{ee} \quad (1.6)$$

For simplicity we consider the B-field to be periodic with spacing d in the direction, and having no spacial dependence in the transverse (x-y) plane . Hence it will be described by

$$B_{0\perp} = B_0 (e_x \cos K_0 z + e_y \sin K_0 z) \qquad (1.7)$$

where $K_0 = 2\pi / d$.

We note that He is invariant under all transformations in the transverse plane and discrete translations in the z direction corresponding ti the periodicity of the field. This results in the Bloch type solution

$$\psi_k (r) = e^{ikr} u_k (z) \qquad (1.8)$$

where u k (z) has the same periodicity as the static magnetic field. It is importatn to note that the function is a four component spinor

$$u_k (z) = \sum_{K=+-K0} C_{Kk},e^{ik.z} \qquad (1.9)$$

where C is a free electron spinor with an energy eigenvalue

$$Ek^2 = h^2 c^2 k^2 + m^2 c^4 \qquad (1.10)$$

To derive the dispersion relation of the free electron laser, the Hamiltonian of the relativistic electron moving in a electron-magnetic field and he magnetic field B may be written as The second quantization formalism for the Hamiltonian for the unpertubed system can be obtained eq,(2.4) , as follows ; as

$$H = \sum_{k\sigma} T_{k\sigma} a^{+}_{k\sigma} a_{k\sigma} + \sum k_{q\lambda} h\omega_{q\lambda} b^{+}_{q,\lambda} b_{q,\lambda} \qquad (1.11)$$

where $a^{+}_{k\sigma}$ is the electron creation operator for the electron momentum hk and spin σ, $ak\sigma$ is the corresponding annihilation opertor, and T_{k}, is its energy . The interaction Hamiltonian can be expressed by

$$H_{int}= \sum_{k2k2q;\sigma1,\sigma2} M_{k1k2q,\sigma1,\sigma2\lambda} a^{+}_{q,\lambda}(t) (b^{+}_{q,\lambda} + b_{q,\lambda})$$

$$+ 1/2 \sum_{k1k2k3k4,\sigma1,\sigma2} V_{k1k3k4k2,\sigma1,\sigma2\sigma2\sigma1}(t)\, a^{+}_{k3,\sigma2} (t) a_{k4,\sigma2}(t) a_{k2,\sigma1}(t) \qquad (1.12)$$

The first term is due to the electron creation-photon interaction with

$$M_{k1k2q,\sigma1,\sigma2\lambda} = -e \sqrt{[4 \pi c\, h /2\, \omega_{q,\lambda} v]}\, (2\pi)2(\delta(k_{1x}-k_{2x} +q_{x})\, \delta(k_{1y}-k_{2y}+q_{y})\, N \sum^{K0}_{K=-K0}$$

$$(\delta(k_{1z} -k_{2z} +q_{z}, K)\, F(^{k1k2}, {}_{\sigma1\sigma2}) \qquad (1.13)$$

In equation (1.25), N the number of spacing, is derived from the

2. The Green function of the photons

The propagation of the phton in this system can be obtained by solving for Green's function of the photon.

The one photon retarded Green function can be wriiten as

$$G_{q,\lambda}(t-t') =\langle\langle A_{q,\lambda}(t); A_{q,\lambda}^{+}(t')\rangle\rangle \quad = -i\, \theta(t-t')\langle[A_{q,\lambda}(t); A_{q,\lambda}^{+}(t')]\rangle \quad (2.1)$$

And where <> denote the statistical average.

Where $[C,D] = CD - DC$,

$\theta(t) =1$ for $t>0$, and $=0$ for $t<0$, (2.2)

$$A_{q,\lambda} = b_{q,\lambda} +b^{+}_{q,\lambda} \quad (2.3)$$

and where <,> denotes the statistical average

To obtain the equation of motion for Green's function , we consider the equations of the motion for the creation and annihilation operators of the photons and electrons from the Hamiltonian defined in eqs. ( ) and ():

$$i\hbar\frac{\partial A_{q,\lambda}(t)}{\partial t}=\left[A_{q,\lambda}(t),H\right]=\hbar\omega_{q,\lambda}(b_{q,\lambda}(t)-b^{+}_{-q,\lambda}(t)) \qquad (3.4)$$

$$=\hbar\omega_{q,\lambda}B_{q,\lambda}(t),$$

ih $\partial A_{q,\lambda}(t)/\partial t$ = $[A_{q,\lambda}(t), H\}$ = $h\omega_{q,\lambda}(b_{q,\lambda}(t)$-$b+_{q,\lambda}(t))$ = $h\omega_{q,\lambda}B_{q,\lambda}(t)$ (2.4)

$$H=H_o+H_{\text{int}}, \qquad (3.5)$$

where H = H_0 + H_{int} (2.5)

$$i\hbar\frac{\partial B_{q,\lambda}(t)}{\partial t}=\left[B_{q\lambda}(t),H\right]=\hbar\omega_{q\omega}A_{q,\lambda}(t)+\sum_{\substack{k_1k_2\\ \sigma_1\sigma_2}}M_{\substack{k_1k_2q\\ \sigma_1\sigma_{2\lambda}}}a^{+}_{k_1\sigma_1}(t)a_{k_2\sigma_1}(t), \qquad (3.6)$$

ih $\partial B_{q,\lambda}(t)/\partial t$ = $[B_{q,\lambda}(t), H]$ = $h\omega_{q,\lambda}A_{q,\lambda}(t)$ +$\sum_{k3q1,\sigma3\lambda1}$ $M_{k2k3q1,\sigma2,\sigma3\lambda1}$a

$a+_{q,\lambda}(t))a_{q,\lambda}(t)$ (2.6)

$$i\hbar\frac{\partial a_{k_1,\sigma_1}(t)}{\partial t}=T_{k_1\sigma_1}a_{k_1\sigma_1}(t)-\sum_{\substack{k_2q_2\\ \sigma_1\lambda}}M_{\substack{k_1k_2q\\ \sigma_1\sigma_2\lambda}}a_{k_2\sigma_2}(t)A_{q,\lambda}(t)+\sum_{\substack{k_2k_3k_4\\ \sigma_2}}V_{\substack{k_1k_3k_4k_2\\ \sigma_1\sigma_2\sigma_2\sigma_1}}(k)a^{+}_{k_3\sigma_2}(t)a_{k_4\sigma_2}(t)a_{k_2\sigma_1}(t), \qquad (3.7)$$

ih $\partial a_{k1,\sigma1}(t)/\partial t$ = $[a_{k1,\sigma1}(t) , H]$ = Ta $_{k1,\sigma1a\ k1,\sigma1}(t)$- +$\sum_{k2q2;\sigma1,\ 2\lambda}$ $M_{k1k2q,\sigma1,\sigma2\lambda}$ a

$a_{q,\lambda}(t)A_{q,\lambda}(t)$ +$\sum_{k2k3k4,\sigma2}$$V_{k1k3k4k2,\sigma1,\sigma2\sigma2\sigma1}(t)$ $a+_{k3,\sigma2}(t)a_{k4,\sigma2}(t)a_{k2,\sigma1}(t)$ (2.7)

$$i\hbar\frac{\partial a^{+}_{k_2\sigma_2}}{\partial t}=-T_{k_2\sigma_2}a^{+}_{k_2\sigma_2}(t)-\sum_{\substack{k_1q\\ \sigma_1\lambda_1}}M_{\substack{k_1k_2q\\ \sigma_1\sigma_2\lambda}}a_{k_1\sigma_1}(t)A_{q,\lambda}(t)+\sum_{\substack{k_1k_3k_4\\ \sigma_2}}V_{\substack{k_1k_3k_4k_2\\ \sigma_1\sigma_2\sigma_2\sigma_1}}(k)a^{+}_{k_3\sigma_2}(t)a^{+}_{k_2\sigma_1}(t)a_{k_4\sigma_2}(t). \qquad (3.8)$$

ih $\partial a+_{k2,\sigma2}$ (t)/∂t = [a+$_{k2,\sigma2}$ (t) , H] = -Ta+$_{k2,\sigma2}$(t) +$\sum_{k1q;\sigma1,\lambda1}$ M $_{k1k2q,\sigma1,\sigma2\lambda}$ a+$_{kqk1\sigma1}$(t)A$_{q,\lambda}$(t)

+$\sum_{k1k3k4,\sigma2}$V $_{k1k3k4k2,\sigma1,\sigma2\sigma2\sigma1}$(k) a+$_{k3,\sigma2}$ (t)a+$_{k2,\sigma1}$(t)a$_{k4,\sigma2}$(t) (2.8)

$$-\hbar^2\left(\frac{\partial^2}{\partial t^2}+\omega_{q\lambda}^2\right)G_{q\lambda}(t-t')=2\omega_{q\lambda}\hbar^2\delta(t-t')+\omega_{q,\lambda}\hbar\sum_{\substack{k_1k_2\\ \sigma_1\sigma_2}}M_{\substack{k_1k_2q\\ \sigma_1\sigma_2\lambda}}G_{k_1\sigma_1,k_2\sigma_2;q\lambda}(t-t'), \qquad (3.9)$$

-h2 (∂^2(t)/∂t^2 + $\omega_{q,\lambda}{}^2$) G (t-t') = 2 $\omega_{q,\lambda}$h^2 kδ(t-t') + $\omega_{q,\lambda}$h $\sum_{k1k2\sigma1;\sigma2}$ M $_{k1k2q,\sigma1,\sigma2\lambda}$ G$_{k1\,\sigma1k2,\sigma1;q\lambda}$(t-t'), (2.9)

$$G_{k_1\sigma_1k_2\sigma_2,q\lambda}(t-t')=\left\langle\left\langle a^+_{k_1\sigma_1}(t)a_{k_2\sigma_2}(t);A^+_{q\lambda}(t')\right\rangle\right\rangle. \qquad (3.10)$$

where G$_{k1\,\sigma1k2,\sigma1;q\lambda}$(t-t')= << , a+$_{k1,\sigma1}$ (t)a$_{k2,\sigma2}$(t); a+$_{k3,\sigma2}$ (t) A$_{q,\lambda}$+(t')>> (2.10)

To solve eq.(2.9), the equation of motion for the Green's function defiend in eq.(2.10) is obtained by differentiating it with respect to time t as sollows;

$$i\hbar\frac{\partial G_{k_1\sigma_1k_2\sigma_2;q\lambda}(t-t')}{\partial t}=(-T_{k_1\sigma_1}+T_{k_2\sigma_2})G_{k_1\sigma_1k_2\sigma_2;q\lambda}(t-t')$$

$$+\sum_{\substack{k_3q_1\\ \sigma_3\lambda_1}}M_{\substack{k_2k_3q_1\\ \sigma_2\sigma_3\lambda_1}}\left\langle\left\langle a^+_{k_1\sigma_1}(t)a_{k_3\sigma_3}(t)A_{q_1\lambda_1}(t);A^+_{q,\lambda}(t')\right\rangle\right\rangle-\sum_{\substack{k_3q_1\\ \sigma_3\lambda_1}}M_{\substack{k_3k_1q_1\\ \sigma_3\sigma_1\lambda_1}}\left\langle\left\langle a^+_{k_3\sigma_3}(t)a_{k_2\sigma_2}(t)A_{q,\lambda}(t);A^+_{q,\lambda}(t')\right\rangle\right\rangle$$

$$+\sum_{\substack{k_3k_4k_5\\ \sigma_3k}}V_{\substack{k_5k_3k_4k_1\\ \sigma_1\sigma_3\sigma_3\sigma_1}}(k)\left\langle\left\langle a^+_{k_5\sigma_1}(t)a^+_{k_3\sigma_3}(t)a_{k_1\sigma_3}(t)a_{k_2\sigma_2}(t);A^+_{q\lambda}(t)\right\rangle\right\rangle$$

$$-\sum_{\substack{k_3k_4k_5\\ \sigma_3k}}V_{\substack{k_2k_3k_4k_1\\ \sigma_2\sigma_3\sigma_3\sigma_2}}(k)\left\langle\left\langle a^+_{k_1\sigma_1}(t)a^+_{k_3\sigma_3}(t)a_{k_4\sigma_3}(t)a_{k_5\sigma_2}(t);A^+_{q,\lambda}(t)\right\rangle\right\rangle. \qquad (3.11)$$

ih ∂G$_{k1\sigma1k2\sigma;q\lambda}$ (t-t')/∂t = (-T$_{k1\sigma1}$+ T$_{2\sigma k2}$) G$_{k1\sigma1k2\sigma;q\lambda}$(t-t')

$+\sum_{k3q1,\sigma3\lambda1} M_{k2k3q1,\sigma2,\sigma3\lambda}1 << a+_{k1\sigma1} (t)a+_{k2\sigma2} (t) A_{q1,\lambda}(t); A_{q,\lambda}^{+}(t') >>$

$-\sum_{k3q1,\sigma3\lambda1} M_{k3k1q1,\sigma3,\sigma13\lambda1} << a+_{k3\sigma3} (t)a+_{k2\sigma2} (t)\ _{Aq1,\lambda}(t);\ _{Aq,\lambda}+(t') >>$

$+\sum_{k3,k4,k5,\sigma3,k} V_{k5k3k4k1\sigma1\sigma3,\sigma3,\sigma1}(k)<< a+_{k5\sigma1} (t)a+_{k3\sigma3} (t)\ a_{k4\sigma3} (t)a_{k2\sigma2} (t); A_{q,\lambda}+(t') >>$

$-\sum_{k3,k4,k5,\sigma3,k} V_{k2k3k4k1\sigma2\sigma3,\sigma3,\sigma2}(k)<< a+_{k1\sigma1} (t)a+_{k3\sigma3} (t)a_{k4\sigma3} (t)a_{k5\sigma2} (t); A_{q,\lambda}+(t') >>$

(2.11)

By successive differentiation of the Green's function with respect to t, hierarchy of equations is obtained. In order to close the hierarchy equation, we approximate the higher order Green's function by expressing it in terms of the low order Green's function. The de0coupling of the higher order Green's function is carried out as follows:

$<< a+_{k1\sigma1} (t)a+_{k3\sigma3} (t)\ A_{q1,\lambda}(t);\ _{Aq,\lambda}+(t') >>\cong$

$\delta_{k1,k3}\delta_{\sigma1,\sigma3}n_{k1,\sigma1}\delta_{q1;q}\ \delta_{\lambda1;\lambda}<< A_{q1,\lambda}(t); A_{q,\lambda}+(t') >>$ (2.12)

where $n_{k1,\sigma1}=< a+_{k1\sigma1} (t)a+_{k3\sigma3} (t)>$ is the density of electrons with momentum h k_1 and spin σ_1.

$<< a+_{k5\sigma1} (t)a+_{k3\sigma3} (t)a_{k4\sigma3} (t)a_{k2\sigma2} (t); Aq,\lambda+(t') >> \sim$

$\delta_{k5,k2}\delta_{\sigma1,\sigma2}n_{k2,\sigma1}<< a+_{k3\sigma3} (t)a+_{k4\sigma3} (t); A_{q,\lambda}+(t') >>$

$-\delta_{k5,k4}\delta_{\sigma1,\sigma3}\ n_{k4,\sigma1}<< a+_{k3\sigma3} (t)a+_{k2\sigma2} (t); A_{q,\lambda}+(t') >>$

$-\delta_{k3,k2}\delta_{\sigma3,\sigma2}n_{k1,\sigma1}<< a+_{k3\sigma3} (t)a+_{k2\sigma2} (t); A_{q,\lambda}+(t') >>$ (2.13)

&&&&&&&&&&&&&&&&&&&&&&&

The first term on the r. h.s. of eq.(2.13) contribute to the direct Coulomb interaction; the second and third terms contribute to the exchange Coulomb interaction.

The fourier components of the equation of motions for the Green's function (2.13) $G_{k1\sigma1k2\sigma;q\lambda}(\omega)$ is thus obtained as

$(h\,\omega+T_{k1\sigma1} - T_{k2\sigma2})\, G_{k1\sigma1k2\sigma;q\lambda}(\omega) =$

$\sum_k 4\pi e^2/k^2[n_{k2\sigma2}\, H(k1, k2,k, \sigma1) - n_{k1\sigma1}\, H(k1, k2,k, \sigma2)],\ x\sum_{k3,k4} H^*(k3, k4,-k, \sigma3)$

$G_{k3\sigma3k4\sigma3;q\lambda}(\omega) + (n_{k2\sigma2} - n_{k1\sigma1})\, M_{k2k1q;\sigma2\sigma1\lambda\, G q\lambda}(\omega)$. (2.14)

Solving eq.(2.14) and substituting it into eq. (2.9), we obtain the equation of motion for photon Green's function:

$[[\, h^2(\omega^2-\omega_{q,\lambda}^2) - h\omega_{q,\lambda}\{\ \sum_{k3,k4\ \sigma1\sigma2} M_{k1k2q;\sigma1\sigma2\lambda}(h\,\omega+T_{k1\sigma1} - T_{k2\sigma2})^{-1}$

$\sum_k [H(k1, k2,k, \sigma1)\, n_{k2\sigma2} - H(k1, k2,k, \sigma2) n_{k1\sigma1}],\ x$

$(4\pi e^2/k^2))[\, 1 + 4\pi e^2/k^2 \sum_{k3k4\ \sigma3} [n_{k4\sigma3} - n_{k3\sigma1}]\,(h\,\omega+T_{k1\sigma1} - T_{k2\sigma2})^{-1}$

$|\, H(k1, k2,k, \sigma2)|^2\,]^{-1}/\sum_{k3,k4\sigma3} H^*(k3, k4,-k, \sigma3)\, M_{k4k3q,\ \sigma3\sigma3\lambda xx}(n_{k2\sigma2}-n_{k1\sigma1})$

$(h\,\omega+T_{k1\sigma1} - T_{k2\sigma2})\text{-}1 + \sum_{k1,k2q;\sigma1\sigma2\lambda}|\, M_{k4k3q,\ \sigma3\sigma3\lambda x}||\, 2(n_{k2\sigma2} - n_{k1\sigma1})\,(h\,\omega+T_{k1\sigma1} - T_{k2\sigma2})\text{-}1]]$

$G_{\lambda q}(\omega) = h^2\omega_{q\lambda}/\pi$ (2.15)

The term [ ] on the r.h.s. of eq. (2.15) account for the Coulomb shielding factor for the photon-electron interaction . In the case of a weak B –field, the primal process (K=0) is predominant in the Coulomb interaction, and H (k1, k2,k, σ2) can be approximated by the δ function, δ(k1-k2+k).

$\omega^2 - q^2c^2 - \omega^2 p/2\gamma\, [1/4(\, hq^2/m\gamma)^2/(\,\omega - qV)^2 - 1/4(\, hq^2/m\gamma)^2\,]$

$= \omega^2_p\, \omega^2_p/4\gamma^3 \sum_{k=+-k0}(((q+K)/K)^2[(\omega-(q+K).Vb)^2 - 1/4(\, h(q+K)/m\gamma)^2 + \omega^2_p/\gamma]^{-1}$

((4.17)

$$[\omega^2 - q^2c^{\,2} - \omega^2{}_p/\gamma] =$$

$$[1/2\omega^2{}_p\, \omega^2{}_{ce} /\gamma^5 (q+K_0)^2/K_0)^2 \,[(\omega-(q+K_0\,).V_b)^2 - \omega^2{}_p /\gamma^3 \,(\,1+3(q+K_0)^2)\lambda D_2\,])]^{\,-1} \qquad (2.18)$$

3. Classical Theory

Although I formulated the quantum mechanical theory for the free electron laser, it was limited to a low intensity laser because of the approximation used to decouple the higher-order correlated function of Eq.(2.12). Hence, it could not be applied for the large intensity amplification of the laser, such as the self amplified spontaneous emission (SASE) mode, which produces a high-intensity laser from the small noise signal or input signal.

To formulate a theory for a high intensity laser, the higher correlated function should not be decoupled as in eq. (2.12). Further, the equation of the motion of the high-order correlated Green function should be obtained by differentiating it the same way as in Eq. (2.9). This differentiation creates a correlated function of a much higher order. By successively differentiating them, a correlated function which is of a higher order than the previous one can be derived. Since the higher correlated function becomes smaller as the order increases, by decoupling the higher order's functions as products of the lower correlation ones, similar to eq. (2.12) succesive simultaneous equations can be closed. However, it is difficult to solve the simultaneous equations analytically; therefore a numerical method using a computer might be required, although it poses the problem how to integrate these continuous functions.

To obtain the closed form of the correlation function, the use of the classical description for the EM field greatly simplifies the formula. When the laser intensity is high, and the number of photons associated with this laser intensity becomes large, their photons are not correlated, such that their phases are randomly distributed, and the EM field can be treated as classical one. Although the photons are not correlated each other, the high -intensity laser affects to the trajectory of electrons, the laser produced from the emission from the electrons is very much affected by this high-intensity laser. In analyzing electron plasma under such a high -intensity laser field, I derived the Green's function of electrons by treating the EM wave as the classical one. In this formalism without a lengthy integration of the simultaneous equation, as used in the above, the intense laser field with multiple of the frequency are simply calculated from the formula of the electron's correlated function,. However, the highly correlated photon is not taken into account this classical formula, and the delicacy of the coherence due to quantum effects is also totally discarded.

When the EM is treated as classical field A(t), the wave function of the electron can be expressed as

$\psi^{(i)}(r,t) = c^{(i)} \exp(ik^{(i)} r - i/(2m^{(i)}) \int_0^t | k^{(i)} + q^{(i)} A(t')|^2 dt$ (3.1)

The total Hamiltonian H is expressed as

$H = \sum_s \{ \sum_s (1/(2m)(k^{(i)} + q^{(i)} A(t))^2 - EF^{(i)}) + 1/2 \sum_{s,t} V^{(i,i)} (r_s - r_t)) \} + \sum_{s,t}(i \neq j) \sum_{s,t} V (r_s - r_t))$ (3.2)

where Vs are the Coulomb interaction hamiltonians.

In this Hamitonian, hamitonian of electron is expressed by $1/(2m)(k^{(i)} + q^{(i)} A(t))^2$ which include the

classical vector field A(t)

The density operator of the i-th particle is expressed in the second quantization formalism as

$\rho^{(i)}(r,t) = \psi^{*(i)}(r,t)\, \psi^{(i)}(r,t) = \sum_{k1,k2\ ak1}{}^{(i)} \exp(ik^{(i)}{}_1\ r)\, a_{k2}\ (i) \exp(ik^{(i)}{}_2\ r)$ (3.3)

The Fourier transformation of the density correlation function operator <ρ(i)(r,t) , ρ(i)(r,t) > can be expressed by

$\iint dr\, dr' <\rho^{(i)}(r,t), \rho^{(i)}(r,t)> \exp(ik^{(i)}(r-r')) = \sum_{k1,k2\sigma1\sigma2} <a^*_{k1\sigma1}{}^{(i)}(t)\, a_{k1\sigma1}{}^{(i)}(t) a+_{k2+k\sigma2}{}^{(j)}(t') a_{k1\sigma 2}{}^{(j)}(t')>$

(3.4)

Fourier Transformation of Green's functions

$G_{k1\sigma1}{}^{(i,j)}(k, t-t') = = \sum_{k2,\sigma2} <<a^*_{k1\sigma1}{}^{(i)}(t)\, a_{k1\sigma1}{}^{(i)}(t) a+_{k2+k\sigma2}{}^{(j)}) (t') a_{k2\ \sigma2}{}^{(j)}(t')>>$ (3.5)

and the vector potential A(t) is sum of many modes as

$A(t) = \sum_s [A_{xs} \cos(\omega_s t + \theta_s) + A_{ys} \sin(\omega_s t + \theta_s)]$ (3.62)

then, Fourier Transformation of Green's functions $G_{k1\sigma1}{}^{(i,j)}(k, E)$ can be obtained as

$G_{\sigma1}{}^{(i,j)}(k, E+\sum_s k_s \omega_s) = \sum_{k1} G_{k1\sigma1}{}^{(i,j)}(k, E+\sum_s k_s \omega_s)$

$= \quad \{\Pi_s\sum_{Ls}J_{L-Ns}\,J(z_s{}^{(i)}{}_s)\,\sum_{Ms}J_{L-Ms}\,J(z_s{}^{(i)}{}_s)\,\exp[-i(\theta_s - -\Delta_s)(N_s - -M_s)]\}$

$\{\,L_{\sigma 1}{}^{(i,)}(k, E+\sum_s k_s\omega_s)\,\delta_{ij} + (4\pi/k^2)\,q^{(i)}L^{(i)}(k, E+\sum sk_s\omega_s)\sum_{j'\neq i}(q^{(j')}\;\; G^{(i,j)}(k, E+\sum_s k_s\omega_s)$

$+- (4\pi/k^2)\,q^{(i)2}\,L_{\sigma 1}{}^{(i)ex}(k, E+\sum sk_s\omega_s)G_{\sigma 1}{}^{(i,j)}(k, E+\sum sMs\omega_s)\} \quad (3.7)$

where

$L_{k1\sigma 1}{}^{(i)}(k, E) = (n_{k1\sigma 1}{}^{(i)} - n_{k2\,\sigma 2}{}^{(i)}) / (E + T_{k1\sigma 1}{}^{(i)} - T_{k2\,\sigma 2}{}^{(i)})$

$L_{\sigma 1}{}^{(i)}(k, E) = \sum_{k1}L_{k1\sigma 1}{}^{(i)}(k, E) \quad L^{(i)}(k, E) = \sum_{\sigma 1}L_{\sigma 1}{}^{(i)}(k, E)$

$L_{\sigma 1}{}^{(i)ex}(k, E) = [1/k^2\sum_{k1}1]^{-1}\sum_{k1k2}1/(k_2-k_1)^2\;\; L_{k2\sigma 1}{}^{(i)}(k, E) \quad (3.8)$

4. Perturbation Theory

One way to save partially the quantum effect on the laser is to use the perturbation method by treating a vector potential A_{em} associated with EM field as sum of a classical field A_c and the A_p. By treating A_p as the operator-expressed creation and annihilation of the photons, the quantum effects on the high-intensity laser can be analyzed as in the first paper, where the vector potential A was composed of the wiggler field A_w and the photon fields. By adding the classical vector potential Ac to the wiggler vector potential,A_w , we can deal with high intensity laser in a similar way as adopted the first paper.

5. Coherent , Squeeze, and Super-Radiant Theories

Another way to deal with this problem is to use the coherent state description which gives the sound foundation of the classical formula.

The coherent state is defined as

$|\alpha> = \exp(-|\alpha|^2/2)\sum_{n=0\text{ to }\inf}\alpha^n/\sqrt{n!}\,|n> \quad (5.1)$

Here $|n> = 1/\sqrt{n!}\,(a+)^n|\varphi_0>, \quad |\varphi_0> = |\text{vacume state}> \quad (5.2)$

is the eigen state of th number operator N=a+ a containing light quanta of (k, η). For our consideration it is useful to split a+ and a into a sum of Hermitian opertors i.e.

$a = (u+ip)/\sqrt{(2h)} \quad a+ = (u-ip)/\sqrt{(2h)} \quad (5.3)$

Coherent state $\alpha a >$ is the eigen state of the non-Hermitian annihilation operator a with the complex eigen value $\alpha =(u+ip) /\sqrt{(2h)}$

If this complex eigen value αa , which labels the coherent states runs over the whole complex plane, the coherent states becomes over-complete for the Hilbert space. Such an over-complete sets of the coherent states can not be used for our consideration because they are linearly dependent. However, Bargmann al. and Perelomov proved that subset of the over complete sub-state form a complete set.

This subset is given by

$\{ |\alpha > : \alpha = \sqrt{\pi} (l +im) ; l = 0,+-1, +-2,; m= = 0,+-1, +-2,\ldots\ldots\}$ (5.4)

This fact was originally stated by von Neumann, without proof. Therefore these states are called von Neumann lattice coherent state(VNLCS).

Using the VNLCS state, Toyoda et al [4] provide the sound foundation of the classical formalism for the high intensity photon field.

This discretization introduces the uncertainty principle of quantum formulation, the delicacy of the quantum effect on the theory such as the super radiant state and the squeeze states which is prominent in the quantum theory of the laser can be studied as discussed as the paper on .

—The one of advantage of using these coherent state description is a many modes created by the EM field can be formulated without difficulty. Further more another transition associated with the super-radiant and between squeezed state can be formulated by including it in the eigen values α,

The squeezed states which might be created by the free electron laser is described by the u or p in the Eq.(5.3)

As mentioned in the state can be expanded by including the these squeeze or super –radiant state, we can formulate the free electron theory which has effect of the these states.

This coherent state is over complete state and the some dicretalization is needed to reduces the over-completeness to the conventional completeness. Here the uncertainty of the quantum description is comes in.

Reference

[1] Hiroshi Takahashi " Theory of the free elctron laser Physica 123 C (1984) 225-237 North-Holland Amsterdam

[2] T. Kwan., J.M.Dawson and A.T. Lin. Phys. Rev. 16A (1977)

[3]Hirshi Takahashi "The electron plasma interacting with high intensity EM field".

[4]
&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&&

$$

The detail discussion of the

As discussed in the use of the coherent state description, the state vector expressing the EM field can be expanded by the coherent state.

In the first paper, vector potential A is the sum of the static wiggler field plus the radiation field, and the wave function of the electron was determined from the static wiggler field, but when the intensity of laser is high this will affect the wave function of the electrons as shown in the second paper. By taking into account this the Green's function of the electrons as obtained as shown in the eq. ( ). Similar approach can be applied to obtain the Green's function of the photon. However, the formula becomes rather complicated to calculate this analytically. But the equation of the motion of the perturbed part of photon can be obtained by numerically using the computer. But due to the imaginarly components involved it requires rather long computer time. And also numerical calculation need the discreatalization of the continuous distribution function, This might have the instability of the calculation, and also the results depend on the segmentation of he continuous function.

The electron's motion is affected as change of the moments of electron as

K + q A(t') and The equation of motion for the electron creation operator is affected as the change of he electron' momentum.

Our interest is photon, thus we can not use this approximation directly, but assuming that the vector field A which correspond to the electro magnetic field is expressed as the sum of the classical components and the photon components

A= Ac +Ap

We can photon part of Ap can be expressed as the creation and annihilation operators shown in the above formula. The equation of motion for these operators can be treated as before the dispersion relation of the photon affected by the high intensity classical field can be obtained. Since the electron motion is affected by the high intensity laser field, the multi mode with multi- mode frequency can be formulated also that the effects of the high intensity laser can be obtained.

The other way of high intensity laser is a use of the coherent sate, which can be more neatly the high intensity laser than the classical one. This coherent state is defined as the eigen state of the annihilation operator for their basic function, thus it is not completely quantum mechanical one

$Ih\ \partial\psi/\partial t = [H, .\ \psi\ \}$

Our system composed of the electrons interacted with the EM field are express as the

Sum of $\psi_{e} * \psi_{em}$
And $\psi_e = \sum_j$ of $\alpha a_j\ \psi_{ej}$

And $\psi_{ew} = \sum_j$ of $\beta_j\ \psi_{em\ j}$

α j and beta j are respectively the amplitude of the electon eigen state of the j and the EM eigen state of electro magnetic eighen state j

In the quatum mechanical formula the se amplitutde are opertors and the Total Hamiltonian of the system are expressed as the

H total =H electron + H EM + H inter

The last term of hamiltonian are came from the staic interaction of the Coulomb interaction of two electrons kl

And it is expressed as sum of the kl 1/r kl

Where r^{kl} is the distrance between kand l electons

The coherent state of the photon is the eigen state of photon which state are eigen state can be secribed as the accwith eigen value

For the electron state

The Hamiltonian can be described by the creation and annihilation operators for electron which is affected by the electro-magnetic wave described by the coherent mode and the photon described by coherent mode and the the interaction hamiltonian between these electrons and photon mode , Equation of the motions for these operators can be derived the similar way as the above formalism,

In this formalism the intensity of the some mode of WM wave are taken into account from begging and the electron motion under influence of the this EM wave is described in the hamiltonian, the high intensity of Laser field can be properly described. The problem of the over completeness of the coherent sate was reduced to the completeness by dicretizing the continuous parameter α. Where the random phase approximation is used which is not exact quantum mechanical some de-coherency is introduced to treat the phenomena as he classical one.

When more accurate formula can be obtained bay refining he discretization mesh size, but it becomes very tedious calculation. Some randomization is needed to get answer.

Although the EM wave is descried as the coherent state, in the above formula. By adding the super radiant states described by Dick the transition of the super radiant transition can be formulated and also, the expansion by the squeeze state mode, we can formulate these transition.

but the coherent effect which came out from the quantum formalism can not be described and the theory which can take into account coherent state should be used Fr this formalism. It has been studied in the squeeze state for free electron laser, this state is squeezed in the phase. Special photons which is a some phase of the electron magnetic wave are peaked. the formalism should be describe the squeezed state.

The other one is the Dick’s super-radiant state which has a coherently coupled with in the

Super radiant transition can be happened and the high intensity monochromatic radiation can be created. This radiation of the free electron laser can not be described by the classical formalism. The quantum mechanical formula using the second quantization method will be useful for this description.

Tadashi Toyoda and Karl Wildermuth " Charged Scrodinger paticl in a c-number radiation field. Phys. Rev D 22, 2391. (1989)

Coherent State description has the problem of the over-completeness, The use of the complete subset of of coherent state of Bargmann et al and Perelomov (called VNLCS), we can describe the radiation field in semi classically.

Two times green function method developed by Prof. Matsubara for solid physics theory
Applied by authors Physica
Compared with the classical formula by Kwan

High intensity laser

By using the spontaneous amplification , we can create the high intensity laser from small input signal, our formalism is limited in the low number of the moment of the correlation function. The obtained results can be applied only for small intensity laser, and very high amplification of the small signal input can not be treated. In order to get the high intensity alser application, the higher moment of correlation function should not be discarded, the whole simultaneous equation including the high moments of the coorelattion should be treated.

By using the two times green function method, we can formalate for the high intensity case, howeveer, the higher order term should be involeved, the formula becomes very complicated, for the low intensity case, the 3rd order term are decoupled by approximation of the

However, it becomes more complicated to formulate this process, we use the some approximation for this one.

When the intensity of the laser becomes high, the classical approximation provide good approximation. As discussed later, the use of the coherent state expanstion justify the this approximation.
When the laser intensity is high the electro- magnetic field created by laser affects of the motion of the electron and the track of the electron through the wiggler magnetic field will be influenced by the strong magnetic field. To treat this effect, we can formulate the free electron laser theory using the classical description of the laser. I formulated this in the paper of the .

“ H. Takahashi “ Interaction between a plasma and a strong electro-magnetic wave, Physica 98C (1980) 313-324

The formula obtained shows clearly the cleation of the higher frequency excitation

The high harmonic of the laser can be described as the

Hamiltonian of Fel

The Hamiltonian of the free-electron laseer can be described by the Equation (1)

Where

This Hamitonian can be descibed the using the second quantiazation method
Second quantization

Dispersion equation for low intensity laser

Electro-phton interaction in the high intensity laser

Formulated equation of motion in the high intensity elctro magnetic wave aaaa9 aprroximated classical field0

FEL in high intensity laser

Self Amplification

Many photon in the high intensity laser
Classical formulation
Coherent state description
Quantum effect can be described

Various description of photn field

Coherent state

Many modes

Dick's super radiant state

Behaviour of laser intensity change can be described by these states. sjcAe

= ω2p ω2p /4γ3 ∑k=+-k0(((q+K)/K)^2[(ω-(q+K).Vb)^2 - 1/4(h(q+K)/mγ)^2 +ω2p /γ]-1

$(T_{k1\sigma1} - T_{k2\sigma2})^{-1}\sum_k$ [H (k1, k2,k, σ1) $_{nk2\sigma2}$ - H (k1, k2,k, σ2) n $_{k1\sigma1}$], x $(4\pi e^2/k^2)$)[1 +$4\pi e^2/k^2$ $\sum_{k3k4\ \sigma3}$

x ++T $_{k1\sigma1}$ –T $_{k2\sigma2}$) G $_{k1\sigma1k2\sigma;q\lambda}(\omega)$ = $\sum_k$ $4\pi e^2/k^2$[n $_{k2\sigma2}$ H (k1, k2,k, σ1)- n $_{k1\sigma1}$ H (k1, k2,k, σ2)], x$\sum_{k3,k4}$ H* (k3, k4,-k, σ3) G $_{k3\sigma3k4\sigma3;q\lambda}(\omega)$ +(n $_{k2\sigma2}$- n $_{k1\sigma1}$) M $_{k2k1q;\sigma2\sigma1\lambda}$ G $_{q\lambda}$ (ω).

$a^+_{k5\sigma1}$ (t)$a^+_{k3\sigma3}$ (t) a $_{k4\sigma3}$ (t)a $_{k2\sigma2}$ (t); A $_{q,\lambda}{}^+$(t') >>

+∑ M $_{k2k3q1,\sigma2,\sigma3\lambda1}$ << $a^+_{k1\sigma1}$ (t)$a^+_{k2\sigma2}$ (t) A $_{q,\lambda}$(t); A $_{q,\lambda}{}^+$(t') >>

where

$$a^+_{k_1\sigma_1}$$

$a^{+}_{k_1\sigma_1}$

is the electron creation operator for electron momentum hk and spin   sigma , aksig is the corresponding annihilation operator, and T is its energy.

The interaction Hamilltonian ca be expressed by

The first term is due to the electron-photon interaction, with

In eq. (2.15), N, is the number of spacing, is derived from the expression

Where K = 2 pin/d (n integer), d is the periodicity of the B field and v is the volume of the system.
The function F-K is given by

The Green function of the photon

The propagation of the photon in this system can be obtained by solving for Green's function of the photon. The dispersion relation of the photon is derived from the equation of motion for Green's function

The one –Photon retarded Green function can be written as

Where

And where < > denotes the statistic average.

To be obtain the equation of motion for Green's function, we consider the equation of motion for the creation and annihilation operators of photon a's and electrons from Hamiltonian defined in the equation and   ;

where

The equation of motion for the Green's function G (t-t') is obtained from

where

To solve eq.(2.9), the equation of motion for the Green's function defined in eq( ) is obtained by differentiating it with respect of time t as follows:

By successive differentiation of the Green's function with respect to t, hierarchy of equations is obtained. In order to close the hierarchy equation, we approximated the higher order Green's function by expressing it in terms of the low order Green's function. The decoupling of he higher order Green's function is carried out as follows:

The first term on the r.h.s. of eq. () contribute to the direct Coulomb interaction; the second and third terms contribute from the exchange Coulomb interaction. In the case of low density of the electron beams. The exchange terms are very small compared with the direct term, so that these terms can be neglected. The Fourier components of the equation of motion for the Green's function G () is thus obtained from

Solving eq.( ) and substituting into eq.( ), we obtain the equation of motion for the photon Green's function:

The term [ ] on the r,h.s. of eq.( ) accounts, for he Coulomb shielding factor for the photo- electron interaction. In the case of a weak B field, the normal process (K=0) is predominant in the Coulomb interaction. And H ( ) can be approximated by the delta function , delta (k_).

In this formalism, the vector potential described for EM field is treated as the classical field.

The electrons beams are influenced by the high intensity EM field, and instead of the Green's function of eq(2.12) should not be treated as the de-coupling the equation of the motion for this Green's function should be taken into accounted, as described in the above the simultaneous equation should be solved for this high intensity cases. Thus for the highly correlated laser the electron wave function is expressed as the sum of the plane wave shown

as $$ih\frac{\partial A_{\mathbf{q},\lambda}(t)}{\partial t}=[A_{\mathbf{q},\lambda}(t),H]=h\omega_{\mathbf{q},\lambda}(b_{\mathbf{q},\lambda}(t)-b^{+}_{-\mathbf{q},\lambda}(t))$$

$H]$

$$= h\omega_{\mathbf{q},\lambda} B_{\mathbf{q},\lambda}(t)$$

$$H = H_0 + H_{\text{int}}$$

where

∑i

j k

$$ih\frac{\partial B_{\mathbf{q},\lambda}(t)}{\partial t} = [B_{\mathbf{q},\lambda}(t), H] = h\omega_{\mathbf{q},\lambda} A_{\mathbf{q},\lambda}(t) + \sum_{\substack{k_1 \; k_2 \\ \sigma_1 \; \sigma_2}} M_{k_2 k_1 k_2} M \;{}_M \; M_{\substack{k_1 \; k_2 \; \mathbf{q} \\ \sigma_1 \; \sigma_2 \; \lambda}} a^+_{k_1\sigma_1} a^+_{k_1\sigma_1} a_{k_1\sigma_1}$$

ih ∂ak1,σ1 (t)/∂t = [ak1,σ1 (t) , H] = Tak1,σ1ak1,σ1(t)- +

∑k2q2;σ1, 2λ M k1k2q,σ1,σ2λ aq,λ(t) a+k3,σ2 (t)ak4,σ2(t)ak2

+∑ k2k3k4,σ2V k1k3k4k2,σ1,σ2σ2σ1(t) a+k3,σ2 (t)ak4,σ2(t)ak2,σ1(t) (2.8)

a+qk3,λ(t))aq,λ(t)ak1,σ1 (t

a $_{k1,\sigma1}$ (t)a $_{k1,\sigma1}$ (t)h$\omega_{q,\lambda}$(b $_{q,\lambda}$(t)-b+ $_{q,\lambda}$(t))

= h$\omega_{q,\lambda}$B $_{q,\lambda}$(t)

A $_{q,\lambda}$(t) A $_{q,\lambda}$(t) A$^{-}_{g,\lambda}$(t)

In the quantum physics the equation of motion of state vector Psi is described as

H = ∑ (c α_ip $_i$- -eA/c) + β_i mc ^{2})

$\sum_{k\sigma} T_{k\sigma}\, a^{+}_{k\sigma}\, a_{k\sigma} + \sum_{kq\lambda} h\omega_{q}\, b^{+}_{q,\lambda}\, b_{q,\lambda}$

$ih\, \partial a_{k1,\sigma1}(t)/\partial t = [a_{k1,\sigma1}(t), H] = Ta_{k1,\sigma1} a_{k1,\sigma1}(t) - +$

$\sum_{k2q2;\sigma1,2\lambda} M_{k1k2q,\sigma1,\sigma2\lambda}\, a_{q,\lambda}(t)\, a^{+}_{k3,\sigma2}(t) a_{k4,\sigma2}(t) a_{k2}$

$+\sum_{k2k3k4,\sigma2} V_{k1k3k4k2,\sigma1,\sigma2\sigma2\sigma1}(t)\, a^{+}_{k3,\sigma2}(t) a_{k4,\sigma}2(t) a_{k2,\sigma1}(t)$ (2.8)

Where A is the sum of the vector potential A_s of the B- field and the vector potential A_s of the electro magnetic wave; $A = A_s + A_r$,

$H \qquad = \sum (c\, \alpha_i p)- -eA/c) + \beta_i\, mc^2)$